

KOSZUL DUALITY FOR QUASI-SPLIT REAL GROUPS

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1. INTRODUCTION

Let $G_{\mathbb{R}}$ be a real semisimple algebraic group and $K_{\mathbb{R}} \subset G_{\mathbb{R}}$ a maximal compact subgroup. We write G, K for the complexifications of $G_{\mathbb{R}}, K_{\mathbb{R}}$, respectively and $\mathfrak{g}, \mathfrak{k}$ for the Lie algebras of G and K . We fix a block \mathcal{M} in the category of Harish-Chandra (\mathfrak{g}, K) -modules and we assume that the generalized infinitesimal central character of modules in \mathcal{M} is regular and integral. We also fix a strong real form for G with underlying real form $G_{\mathbb{R}}$.

Let \check{G} be Langlands dual complex group. Vogan [Vo4] assigned to the above data a (strong) real form of \check{G} and a block $\check{\mathcal{M}}$ in the category of $(\check{\mathfrak{g}}, \check{K})$ -modules with a regular integral infinitesimal central character. [The infinitesimal central character is defined up to translation by the infinitesimal central character of a finite dimensional \check{G} -representation, but the equivalence class of the block is well defined].

We impose an additional assumption that $\check{\mathcal{M}}$ is a *principal block*, i.e., that $\check{\mathcal{M}}$ contains a finite dimensional \check{G} -representation. This implies that $G_{\mathbb{R}}$ is quasi-split.

By Beilinson-Bernstein Localization Theorem we can realize $\check{\mathcal{M}}$ as a category of twisted \check{K} -equivariant D -modules on the flag variety $\check{\mathcal{B}} = \check{G}/\check{B}$. Moreover, since $\check{\mathcal{M}}$ contains a finite dimensional representation, we can (and will) realize $\check{\mathcal{M}}$ as a subcategory of \check{K} equivariant D -modules on \check{G}/\check{B} (i.e. in fact no twisting is needed). Let $\check{\mathcal{D}}$ be the full subcategory in the equivariant derived category $D_{\check{K}}(\check{G}/\check{B})$ consisting of complexes whose cohomology belongs to $\check{\mathcal{M}}$.

The central result of this paper is a *Koszul duality* between the categories $D^b(\mathcal{M})$ and $\check{\mathcal{D}}$. Thus we categorify Vogan character duality in [Vo4] for quasisplit $G_{\mathbb{R}}$ and consequently prove the conjecture by Soergel [So2] in this setting. The present work generalizes results and methods of Soergel's work [So1] which treated the case of a complex group.

In more detail, we establish a certain relationship between the *triangulated* categories on the two Langlands dual sides: the derived category of the block \mathcal{M} for modules with *generalized* integral regular central character for the quasi-split real group $G_{\mathbb{R}}$ and a block in the *equivariant derived category* $D_{\check{K}}(\check{\mathcal{B}})$. Namely, we show that the abelian category \mathcal{M} is equivalent to the category of nilpotent modules over the ring $A = \text{Ext}^{\bullet}(\oplus L_i, \oplus L_i)$, where L_i run over the set of irreducible perverse sheaves in the block of $D_{\check{K}}(\check{\mathcal{B}})$.

By a standard formality argument this implies an equivalence of appropriate graded versions of the triangulated categories. Observe that $D_{\check{K}}(\check{\mathcal{B}})$ can be thought of as a *derived version* of the category of $(\check{\mathfrak{g}}, \check{K})$ -modules with a fixed infinitesimal central character (when tensoring with the base field over the center of the enveloping algebra $U(\mathfrak{g})$ one needs to take into account higher Tor's, see section 5 for details). Thus in general our result does not yield two Koszul dual rings controlling the representation categories on the two sides; however, we do get such a picture in special cases, such as the previously known case of a complex group and the new (to our knowledge) case of the principal block in a split group.

Let us recall the mechanism by which the categorical equivalence yields a *duality* at the level of Grothendieck groups (as already explained in [So2]). By elementary algebra the Grothendieck group of nilpotent A -modules $K^0(A - \text{mod}_{\text{nilp}})$ is dual to

the Grothendieck group of all finitely generated A -modules $K^0(A - \text{mod}_{fg})$, the dual bases in the two groups are provided by the classes of graded irreducible modules and indecomposable graded projective modules respectively. Our result explained above implies that $K^0(A - \text{mod}_{nilp}) \cong K^0(\mathcal{M})$, while the Grothendieck group of the block in $D_{\check{K}}(\check{\mathcal{B}})$ is identified with $K^0(A - \text{mod}_{fg})$ by sending the class of an irreducible $L_j \in D_{\check{K}}(\check{\mathcal{B}})$ to the module $\text{Ext}^\bullet(\bigoplus_i L_i, L_j)$; thus it yields a duality between the two Grothendieck groups.

We finish the introduction by indicating the key idea of our method. Recall that a central role in Soergel's theory is played by the *Soergel bimodules*, which form a full subcategory in the category of coherent sheaves on $\mathfrak{t}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$, where \mathfrak{h} is the abstract Cartan algebra of \mathfrak{g} . An analogous role in our construction is played by the so called *block variety*, $\mathfrak{B} = \mathfrak{a}^*/W'_M \times_{\mathfrak{h}^*/W} \mathfrak{t}^*$, where \mathfrak{a} is a maximal split torus in \mathfrak{g} and $W'_M \subset W$ is a subgroup depending on the block \mathcal{M} . The appearance of the quotient \mathfrak{a}^*/W'_M can be motivated as follows. On the one hand it (or rather its completion at zero) can be realized as the space parametrizing deformations of an irreducible principal series module at the singular central character $-\rho$, which belongs to the image of \mathcal{M} under the translation functor. On the other hand, the ring $\mathcal{O}(\mathfrak{a}^*/W'_M)$ is identified with the equivariant cohomology ring $H_{\check{K}}^\bullet(\check{\mathcal{B}})$. The argument proceeds then by describing the categories of projective pro-objects in \mathcal{M} , as well as the subcategory of semisimple complexes in the block of $D_{\check{K}}(\check{\mathcal{B}})$ as full subcategories in the coherent sheaves on the block variety (or rather its completion at zero). In the case when the group is not adjoint one needs to modify the above strategy by considering coherent sheaves on the block variety equivariant with respect to a certain finite abelian group.

In this paper we use a generalization of the by now classical approach initiated in [So1], [BGS], combining it with combinatorial information from Vogan's work [Vo4]. Ben-Zvi and Nadler [BZN] have proposed a way to approach Soergel's conjectures via geometric Langlands duality. It would be interesting to relate the two approaches. We expect that the relation of our construction to Hodge D -modules (see section 6) may provide a starting point for this line of investigation.

The paper is organized as follows.

In section 2 we recall some combinatorial invariants of our categories. Most of these appear in [ABV], we also use [AdC] as a convenient reference.

In section 3 we give a description of $\check{\mathcal{D}}$ based on the functor of cohomology (derived global sections) which turns out to be faithful on the subcategory of irreducible objects.

In section 4 we provide a description of category \mathcal{M} based on the functor of translation to singular central character which turns out to be faithful on the subcategory of projective (pro)objects.

In section 5 we formally state and prove our Koszul duality result by comparing the two descriptions. We end the paper by section 6 where we present a conjecture on realization of \mathcal{M} in terms of coherent sheaves on the cotangent bundle via Hodge D -modules and explain its relation to our description of \mathcal{M} .

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2. COMBINATORICS OF A BLOCK

In this section we recall the basic facts about Vogan duality [Vo4, ABV, AdC] and its underlying combinatorics in the setting explained in the introduction. The duality involves making certain choices and part of the statement is that those choices can be made consistently. In the discussion below we recall the two sides of the duality and for the precise matching we refer to the references listed above.

2.1. The equivariant side. The combinatorial data related to the block $\check{\mathcal{M}}$ is as follows. Let \check{C} be a maximal torus in \check{K} and $\check{T} = Z_{\check{G}}(\check{C})$ be the maximal torus in \check{G} containing \check{C} . Let $W(\check{K}) = N_{\check{K}}(\check{C})/\check{C}$ be the Weyl group.

We have an involution $\check{\theta}$ on \check{G} with $\check{G}^{\check{\theta}} = \check{K}$. As the torus \check{T} is stable under $\check{\theta}$, it induces an involution of the Weyl group $W = N(\check{T})/\check{T}$. We write $W^{\check{\theta}}$ for the subgroup of W fixed by this involution.

Note that the group \check{K} may be disconnected. We have a homomorphism $W(\check{K}) \rightarrow \pi_0(\check{K})$ which is onto. Its kernel is the group $W(\check{K}^0)$ where \check{K}^0 is the identity connected component in \check{K} . Note that the group $\pi_0(\check{K}) \cong \pi_0(\check{C})$ is a 2-group, i.e., it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$ for some r .

Thus we have

$$W(\check{K}^0) \subset W(\check{K}) \subset W^{\check{\theta}}.$$

Finally, we need a count of the closed \check{K} -orbits on the flag manifold $\check{\mathcal{B}} = \check{G}/\check{B}$.

Let $(\check{K} \backslash \check{\mathcal{B}})_{cl}$ be the set of closed \check{K} orbits on $\check{\mathcal{B}}$. The group \check{W} acts simply transitively on the set of \check{T} fixed points on $\check{\mathcal{B}}$. Hence, by choosing a base point, we can identify the set of \check{T} fixed points on $\check{\mathcal{B}}$ with \check{W} . The \check{W} -action is naturally a right action, which we turn to a left action with w acting via w^{-1} . For an orbit $O \in \check{K} \backslash \check{\mathcal{B}}$ the intersection $O \cap \check{\mathcal{B}}^{\check{T}}$ is either empty or it is a coset of $W(\check{K})$; moreover, the intersection is nonempty if the orbit is closed. Thus we get an injective map $\phi : (\check{K} \backslash \check{\mathcal{B}})_{cl} \rightarrow \check{W}/W(\check{K})$.

Claim 2.1. *The map ϕ identifies $(\check{K} \backslash \check{\mathcal{B}})_{cl}$ with $\check{W}^{\check{\theta}}/W(\check{K})$. The choice of the base point is dictated by the combinatorics in [Vo4].*

Let us now consider the above situation on the level of Lie algebras. Then $\check{\mathfrak{c}}$ is a maximal torus in $\check{\mathfrak{k}}$ and $Z_{\check{\mathfrak{g}}}(\check{\mathfrak{c}}) = \check{\mathfrak{t}}$ is a maximal $\check{\theta}$ -stable torus in $\check{\mathfrak{g}}$. It will be convenient to introduce a separate notation $\check{\mathfrak{h}}$ for the abstract Cartan algebra of $\check{\mathfrak{g}}$.

Recall also that $W = \check{W}$, and so we will often denote the Weyl group just by W . Any fixed point of the $\check{\theta}$ -stable torus $\check{\mathfrak{t}}$ gives rise to an identification between $\check{\mathfrak{t}}$ and $\check{\mathfrak{h}}$ and these identifications are related by the Weyl group action. Hence, there is a canonical identification $\check{\mathfrak{t}}/W = \check{\mathfrak{h}}/W$ and thus we obtain a canonical map $\check{\mathfrak{c}}/W(\check{K}^0) \rightarrow \check{\mathfrak{h}}/W$. In this setting we define the block variety as $\mathfrak{B}_{eq} = \check{\mathfrak{c}}/W(\check{K}^0) \times_{\check{\mathfrak{h}}/W} \check{\mathfrak{h}}$. The group $\mathfrak{S}_{eq} = W(\check{K})/W(\check{K}^0)$ acts on \mathfrak{B} via its natural action on the first factor.

2.2. The quasisplit side. Recall from the introduction that we have a quasi-split real form $G_{\mathbb{R}}$ and a fixed block in the category of (finite length) (\mathfrak{g}, K) -modules which we denoted by \mathcal{M} . All modules in the block have the same generalized infinitesimal central character which is assumed to be regular and integral; thus it corresponds to a uniquely defined dominant weight denoted by λ .

We identify \mathcal{M} with its image under the localization equivalence between the category of (\mathfrak{g}, K) -modules and the category of K -equivariant twisted $D_{\hat{\lambda}}$ -modules on the flag variety $\mathcal{B} = G/B$; here $D_{\hat{\lambda}}$ denotes an infinitesimally deformed twisted differential operators ring corresponding to the line bundle $\mathcal{O}(\lambda)$.

A $(K$ -equivariant) $D_{\hat{\lambda}}$ module is by definition the same as a $(K$ -equivariant) D -module on the extended flag manifold $X = G/U$, where U denotes the unipotent radical of B , which is weakly equivariant with respect to the right action of the universal Cartan $H \cong B/U$ with the generalized eigenvalues of the log monodromy determined by λ .

An irreducible object in that category is of the form $j_{!*}(\mathcal{L})$ where $j : O \rightarrow X$ is the embedding of a $K \times H$ -orbit and \mathcal{L} is an irreducible K -equivariant and H -monodromic local system.

We say that the pair (O, \mathcal{L}) belongs to the block if $j_{!*}(\mathcal{L}) \in \mathcal{M}$. We let $\mathcal{L}_{\mathcal{M}}(O)$ denote the set of local systems on O belonging to the block \mathcal{M} . Let $\text{Irr}(\mathcal{M})$ be the set of irreducible objects in \mathcal{M} , thus we have $\text{Irr}(\mathcal{M}) \cong \bigcup_{O \in K \backslash G/B} \mathcal{L}_{\mathcal{M}}(O)$.

For the rest of this section the orbit O will be the open $K \times H$ -orbit X_0 on X which is the inverse image of the open K -orbit $\mathcal{B}_0 \subset \mathcal{B}$. Recall the maximal torus \tilde{T} of \tilde{G} and let us write T for its dual in G . The torus T is θ -stable. We let $A \subset T$ be the fixed points of the involution $x \mapsto \theta(x^{-1})$, the connected component A^0 is the complexification of the maximal split torus in $G_{\mathbb{R}}$. We let \mathfrak{a} be the Lie algebra of A , thus \mathfrak{a} is the (-1) eigenspace of θ acting in \mathfrak{t} . For the maximally split torus T the real Weyl group $W_{\mathbb{R}} = W(K, T)$ coincides with W^{θ} . Recall also that the fixed points \mathcal{B}^T give rise to K -orbits on \mathcal{B} ; we say that these K -orbits are attached to T . These include the open orbit \mathcal{B}_0 .

We recall, as is rather easy to see, that all blocks have representatives on the open orbit, i.e., that $\mathcal{L}(X_0)$ is not empty. As the action of $K \times H$ is transitive on X_0 , the set $\mathcal{L}_{\mathcal{M}}(X_0)$ is identified with a subset in the set of irreducible representations of the group $\pi_0(\text{Stab}_{K \times H}(\tilde{x})) = \pi_0(\text{Stab}_K(x))$ where $x \in \mathcal{B}_0$ and $\tilde{x} \in X_0$ projects to x . We make this choice so that $x \in \mathcal{B}_0$ is a fixed point of T . The finite group $\pi_0(\text{Stab}_K(x)) \cong T^{\theta}/(T^{\theta})^0$ is a 2-group, i.e., it is isomorphic to a product of a finite number of copies of $\mathbb{Z}/2\mathbb{Z}$.

We will make use of the *cross-action* of W on the set $\text{Irr}(\mathcal{M})$, [Vo2, Definition 8.3.1], [AdC, sections 9 and 14]. We will summarize some properties of the cross action that will be important to us in section 4.3.

In particular, W acts on $\cup \mathcal{L}(O)$ where O runs through the orbits attached to the Cartan T . The subgroup $W_{\mathbb{R}} = W^{\theta}$ of W fixes the open orbit and hence it acts on $\mathcal{L}(X_0)$. We have:

Claim 2.2. *The action of W^{θ} on $\mathcal{L}(X_0)$ is transitive.*

We choose $L \in \mathcal{L}(X_0)$ and set $W_{\mathcal{M}} = \text{Stab}_{W^{\theta}}(L)$. Thus $W_{\mathcal{M}}$ is a subgroup in W^{θ} defined up to conjugacy.

We will also need a comparison between the relevant data for G and for the adjoint group G_{ad} equipped with the compatible real form.

Let $K' \subset G$ be the subgroup generated by K and the center $Z(G)$ of G . Note that the finite group $Z(G) \cap K$ acts on all modules in \mathcal{M} by the same character which we denote by χ . Fix an extension $\tilde{\chi}$ of χ to a character of $Z(G)$. It is easy to see that the category of (\mathfrak{g}, K) modules on which $Z(G) \cap K$ acts by χ is equivalent to the category of (\mathfrak{g}, K') modules where $Z(G)$ acts by $\tilde{\chi}$. We identify \mathcal{M} with its image under that equivalence; localization theorem identifies this category with a subcategory in K' -equivariant $D_{\tilde{\chi}}$ modules. In particular, elements of $\mathcal{L}_{\mathcal{M}}(X_0)$ can be viewed as K' -equivariant local systems on the corresponding orbits.

Let K_{ad} be the fixed points of the involution of G_{ad} compatible with the one fixed on G . Let \tilde{K}_{ad} be the preimage of K_{ad} under the homomorphism $G \rightarrow G_{ad}$. The group $\tilde{K}_{ad}/K' = K_{ad}/\text{Im}(K)$ is abelian, we let $\mathfrak{S}_{mon} = (K_{ad}/\text{Im}(K))^*$ be the dual abelian group.

Let $\mathcal{L}_{\mathcal{M}}^{ad}(X_0)$ be the set of \tilde{K}_{ad} equivariant local systems on X_0 which, viewed as a K' -equivariant local system, belong to $\mathcal{L}_{\mathcal{M}}(X_0)$. For $x \in \mathcal{B}_0$ we have a short exact sequence of abelian groups

$$\{1\} \rightarrow \pi_0(\text{Stab}_{K'}(x)) \rightarrow \pi_0(\text{Stab}_{\tilde{K}_{ad}}(x)) \rightarrow \mathfrak{S}_{mon}^* \rightarrow \{1\}.$$

Thus we have a free action of \mathfrak{S}_{mon} on $\mathcal{L}_{\mathcal{M}}^{ad}(X_0)$, such that $\mathcal{L}_{\mathcal{M}}^{ad}(X_0)/\mathfrak{S} = \mathcal{L}_{\mathcal{M}}(X_0)$.

The cross action lifts to an action of W^θ on $\mathcal{L}_{\mathcal{M}}^{ad}(X_0)$, which is also transitive. We let $W'_{\mathcal{M}}$ be the stabilizer of a point in $\mathcal{L}_{\mathcal{M}}^{ad}(X_0)$ under this action. The considerations above show that we have a short exact sequence of groups:

$$(2.1) \quad \{1\} \rightarrow W'_{\mathcal{M}} \rightarrow W_{\mathcal{M}} \rightarrow \mathfrak{S} \rightarrow \{1\}.$$

Finally, we can define the block variety in this language as $\mathfrak{B}_{mon} = \mathfrak{a}^*/W'_{\mathcal{M}} \times_{\mathfrak{h}^*/W} \mathfrak{h}^*$. The group \mathfrak{S}_{mon} acts on \mathfrak{B}_{mon} via its action on the first factor which comes from the exact sequence (2.1).

2.3. Matching. In the two previous sections we have defined two versions of the block variety \mathfrak{B}_{eq} and \mathfrak{B}_{mon} with actions of \mathfrak{S}_{eq} and \mathfrak{S}_{mon} , respectively. We now make use of the combinatorics of Vogan duality [Vo4] to show that these two definitions coincide. This is a crucial ingredient in our arguments.

In the discussion above we have had defined a set of local systems $\mathcal{L}_{\mathcal{M}}^{ad}(X_0)$. We can also think of these local systems as belonging to a block \mathcal{M}^{ad} of (\mathfrak{g}, K_{ad}) -modules, but to do so we have to move to a different infinitesimal character. To this end we choose an irreducible representation V_μ or highest weight μ such that the center $Z(G)$ acts on V_μ by the character $\tilde{\chi}^{-1}$. We now make use of the translation functor $T_{\lambda \rightarrow \lambda + \mu}$ (which is an equivalence of categories, of course) to translate our set up from the generalized infinitesimal character λ to the generalized infinitesimal character $\lambda + \mu$. After the translation the modules $\mathcal{L}_{\mathcal{M}}^{ad}(X_0)$ belong to a particular block \mathcal{M}^{ad} of (\mathfrak{g}, K_{ad}) -modules. More precisely, we have

$$T_{\lambda \rightarrow \lambda + \mu} \text{ carries the set } \mathcal{L}_{\mathcal{M}}^{ad}(X_0) \text{ to the set } \mathcal{L}_{\mathcal{M}^{ad}}(X_0).$$

After the translation, the blocks \mathcal{M} and \mathcal{M}^{ad} can be compared directly as the modules in \mathcal{M} consist now of certain $(\mathfrak{g}, \text{Im}(K))$ -modules. We have:

Theorem 2.3. *a) There exist compatible isomorphisms:*

$W \cong \check{W}; \check{\mathfrak{t}} \cong \mathfrak{t}^$, sending the involution θ to $-\theta$.*

In particular, we get an isomorphism $\check{\mathfrak{c}} \cong \mathfrak{a}^$.*

b) The isomorphism $W^{\check{\theta}} \cong W^{\theta}$ arising from (a) sends the conjugacy class of the pair of subgroups $W'_{\mathcal{M}} \subset W_{\mathcal{M}}$ to that of the pair $W(\check{K}^0) \subset W(\check{K})$.

This results is easy to extract from the formulation of Vogan duality in [AdC, Theorem 10.3]. We consider the block \mathcal{M} and the corresponding block \mathcal{M}^{ad} for K_{ad} . According to [AdC, Theorem 10.3] these blocks give rise to real forms \check{K} of \check{G} and \check{K}_{sc} of \check{G}_{sc} . The image of $\check{K}_{sc} \rightarrow \check{K}$ is the connected component \check{K}^0 of \check{K} . Furthermore, the irreducible local systems on the open orbit O in our blocks \mathcal{M} and \mathcal{M}^{ad} correspond closed orbits of \check{K} and \check{K}_{sc} , respectively, on \check{B} . The cross action of W^{θ} on the irreducible local systems corresponds to the usual W^{θ} -action on the closed orbits.

Corollary 2.4. *The two pairs $(\mathfrak{B}_{eq}, \mathfrak{S}_{eq})$ and $(\mathfrak{B}_{mon}, \mathfrak{S}_{mon})$ are isomorphic. We will denote this pair by $(\mathfrak{B}, \mathfrak{S})$ from now on.*

3. PRINCIPAL BLOCK IN THE EQUIVARIANT DERIVED CATEGORY

In this section we consider a real form of \check{G} and we write \check{K} for the complexification of the maximal compact subgroup of the real form. We work with the principal block \mathcal{D} in the category $D_{\check{K}}^b(\check{B})$ and with the other notation introduced in section 2.

Recall that $\check{\mathfrak{c}}$ is a maximal torus in $\check{\mathfrak{k}}$ and then $Z_{\check{\mathfrak{g}}}(\check{\mathfrak{c}}) = \check{\mathfrak{t}}$ is a maximal $\check{\theta}$ -stable torus in $\check{\mathfrak{g}}$ and that we write $\check{\mathfrak{h}}$ for the abstract Cartan algebra of $\check{\mathfrak{g}}$.

Recall also that $W = \check{W}$, and so we will often denote the Weyl group just by W .

We begin by considering the case when the group \check{K} is connected. In particular, $H^*(\text{pt}/\check{K}) \cong \mathbb{C}[\check{\mathfrak{c}}]^{W(\check{K}, \check{\mathfrak{c}})}$ and we have a canonical map $\check{\mathfrak{c}}/W(\check{K}) \rightarrow \check{\mathfrak{h}}/W$.

Lemma 3.1. *We have a canonical isomorphism: $H_{\check{K}}^*(\check{B}) = \mathbb{C}[\mathfrak{B}]$.*

Proof. Let us choose a particular Borel \check{B} in \check{G} . This gives us the following diagram on the level of spaces

$$(3.1) \quad \begin{array}{ccc} \check{K} \backslash \check{G} / \mathcal{B} & \longrightarrow & \text{pt} / \check{K} \\ \downarrow & & \downarrow \\ \text{pt} / \check{B} & \longrightarrow & \text{pt} / \check{G}. \end{array}$$

Note that up to homotopy we can replace pt / \check{B} by pt / \check{H} so the diagram itself is independent of the choice of \check{B} .

Passing to cohomology and using the canonical isomorphisms $H^*(\text{pt}/\check{K}) \cong \mathbb{C}[\check{\mathfrak{c}}]^{W(\check{K})}$, $H^*(\text{pt}/\check{G}) \cong \mathbb{C}[\check{\mathfrak{h}}]^W$, and $H^*(\text{pt}/\check{H}) \cong \mathbb{C}[\check{\mathfrak{h}}]$ give us a canonical map

$$\mathbb{C}[\check{\mathfrak{h}}] \otimes_{\mathbb{C}[\check{\mathfrak{h}}]^W} \mathbb{C}[\check{\mathfrak{c}}]^{W(\check{K})} \rightarrow H_{\check{K}}^*(\check{B}).$$

On the other hand, the Serre spectral sequence associated with the upper horizontal arrow in (3.1) degenerates, as it has zero terms in odd degrees. This gives the conclusion. \square

3.1. Global cohomology is fully faithful on simples (statement). Since global equivariant cohomology of a space acts on the cohomology (derived global sections) of any equivariant complex, we get a functor $R\Gamma_{\check{K}} : \mathcal{D} \rightarrow \mathcal{C}oh(\mathfrak{B})$. Let \mathcal{L} be the category whose objects are semisimple complexes in \mathcal{D} and whose morphisms are given by $Hom_{\mathcal{L}}(L_1, L_2) = Ext_{\mathcal{D}}^{\bullet}(L_1, L_2)$.

One of the main results of this section is the following

Theorem 3.2. *The functor $R\Gamma_{\check{K}} : \mathcal{L} \rightarrow \mathcal{C}oh(\mathfrak{B})$ is a full embedding.*

We begin by spelling out some properties of the functor $R\Gamma_{\check{K}}$.

3.2. Components of \mathfrak{B} and closed orbits. The irreducible components of \mathfrak{B} are indexed by $W/W(\check{K})$ and can be described as follows. Let us recall that we have fixed a particular bijection $\phi : (\check{K} \backslash \check{\mathfrak{B}})_{cl} \rightarrow \check{W}^{\check{\theta}}/W(\check{K})$ in 2.1. In particular, we have fixed a closed \check{K} -orbit S_0 corresponding the identity coset. Observe that in this manner the bijection ϕ extends to a bijection between the \check{K} -orbits on $\check{\mathfrak{B}}$ associated to the Cartan $\check{\mathfrak{t}}$ and $\check{W}/W(\check{K})$. Choosing a Borel given by a fixed point of $\check{\mathfrak{t}}$ on the orbit S_0 we obtain an identification $\psi : \check{\mathfrak{t}} \xrightarrow{\sim} \check{\mathfrak{h}}$ which is well defined up to conjugacy by $W(\check{K})$.

From the identification $\psi : \check{\mathfrak{t}} \xrightarrow{\sim} \check{\mathfrak{h}}$ we obtain an embedding $\iota : \check{\mathfrak{c}} \rightarrow \check{\mathfrak{h}}$ which is well-defined up to conjugation by $W(\check{K})$. For an element $w \in W$ the map $\gamma_w : \check{\mathfrak{c}} \rightarrow \mathfrak{B}$, $\gamma_w : x \mapsto (x, w \circ \iota)$ is obviously a closed embedding and its image \mathfrak{B}_w is an irreducible component of \mathfrak{B} . For $w' \in w \cdot W(\check{K})$ we have $\mathfrak{B}_w \cong \mathfrak{B}_{w'}$, thus we obtain a well-defined component $\mathfrak{B}_w \subset \mathfrak{B}$ for any $w \in W/W(\check{K})$. Note that in this manner we have also obtained a particular bijection between the irreducible components of the the block variety \mathfrak{B} and the set of \check{K} -orbits on $\check{\mathfrak{B}}$ associated to the Cartan $\check{\mathfrak{t}}$.

Lemma 3.3. *For $S \in (\check{K} \backslash \check{\mathfrak{B}})_{cl}$ we have*

$$R\Gamma(\mathbb{C}_S[\dim S]) \cong \mathcal{O}_{\mathfrak{B}_{\phi(S)}},$$

where ϕ is as in Claim 2.1.

Proof. As S is isomorphic to the flag variety for \check{K} , we see that $H_K^*(S) \cong \mathbb{C}[\check{\mathfrak{c}}]$. By choosing a $\check{\mathfrak{t}}$ -fixed point on S we obtain a θ -stable Borel and that Borel gives rise to an identification $\psi_S : \check{\mathfrak{t}} \xrightarrow{\sim} \check{\mathfrak{h}}$. This identification is related to the identification ψ by the element $\phi(S) \in W^{\check{\theta}}$. Thus, the map

$$\mathbb{C}[\check{\mathfrak{h}}] \xrightarrow{\Psi_S^*} \mathbb{C}[\check{\mathfrak{t}}] = H_G^*(\check{\mathfrak{B}}) \rightarrow H_K^*(S) = \mathbb{C}[\check{\mathfrak{c}}]$$

gives rise to the map $w \circ \iota : \check{\mathfrak{c}} \rightarrow \check{\mathfrak{h}}$. This gives us the conclusion. \square

3.3. Zuckerman functors and Soergel functors. Recall that for every simple root α we have the corresponding minimal parabolic P_{α} , the partial flag variety \mathcal{P}_{α} and a \mathbb{P}^1 fibration $\pi_{\alpha} : \check{\mathfrak{B}} \rightarrow \mathcal{P}_{\alpha}$. Set $C_{\alpha} = \pi_{\alpha}^* \pi_{\alpha*} : \mathcal{D} \rightarrow \mathcal{D}$. By the decomposition theorem C_{α} acts also on the category \mathcal{L} .

Set $\mathfrak{B}_{\alpha} = \check{\mathfrak{c}}/W(\check{K}) \times_{\check{\mathfrak{h}}/W} \check{\mathfrak{h}}/W_{\alpha}$ where $W_{\alpha} = \{1, s_{\alpha}\} \subset W$, here s_{α} is the simple reflection of type α . We have a degree two map $pr_{\alpha} : \mathfrak{B} \rightarrow \mathfrak{B}_{\alpha}$.

Lemma 3.4. *We have a canonical isomorphism of functors: $\mathcal{D} \rightarrow \mathcal{C}oh(\mathfrak{B})$:*

$$R\Gamma_{\check{K}} \circ C_\alpha \cong pr_\alpha^* pr_{\alpha*} \circ R\Gamma_{\check{K}}.$$

Proof. By functoriality of π_α^* we have a map $R\Gamma_{\check{K}}(\pi_{\alpha*}(\mathcal{F})) = R\Gamma_{\check{K}}(\mathcal{F}) \rightarrow R\Gamma_{\check{K}}(\pi_\alpha^* \pi_{\alpha*}(\mathcal{F}))$. It is clear that this map is compatible with the action of $H_{\check{K}}^*(\mathcal{P}_\alpha) = \mathcal{O}(\mathfrak{B}_\alpha)$. Thus we get a map from the right hand side to the left hand side of the required isomorphism. The direct image of the constant sheaf $\pi_{\alpha*}(\mathbb{C}_{\check{\mathcal{B}}})$ is isomorphic to $\mathbb{C}_{\check{\mathcal{B}}_\alpha} \oplus \mathbb{C}_{\check{\mathcal{B}}_\alpha}[-2]$. It follows that

$$R\Gamma_{\check{K}}(\pi_\alpha^* \pi_{\alpha*}(\mathcal{F})) \cong R\Gamma_{\check{K}}(\pi_{\alpha*}(\mathcal{F})) \oplus R\Gamma_{\check{K}}(\pi_{\alpha*}(\mathcal{F}))[-2] \cong R\Gamma_{\check{K}}(\mathcal{F}) \oplus R\Gamma_{\check{K}}(\mathcal{F})[-2],$$

which implies the above map is an isomorphism. \square

3.4. Closed orbits generate the block. The following statement is standard, see e.g. [LV].

Proposition 3.5. *The objects $\mathbb{C}_S[\dim S]$, $S \in (\check{K} \backslash \check{\mathcal{B}})_{cl}$ generate \mathcal{L} under the action of the functors C_α , direct sums and direct summands.*

We now embark on the proof of Theorem 3.2.

3.5. Beginning of the proof of Theorem 3.2: reduction to closed orbits. We need to show that

$$(3.2) \quad Hom_{\mathcal{L}}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} Hom_{\mathcal{C}oh(\mathfrak{B})}(R\Gamma_{\check{K}}(\mathcal{F}), R\Gamma_{\check{K}}(\mathcal{G}))$$

for $\mathcal{F}, \mathcal{G} \in \mathcal{L}$. Notice that the functor C_α is self-adjoint (up to a shift) and $pr_\alpha^* pr_{\alpha*}$ is self-adjoint. Thus, the validity of isomorphism (3.2) for a given $\mathcal{G} = \mathcal{G}_0$ and all \mathcal{F} implies its validity for $\mathcal{G} = C_\alpha(\mathcal{G}_0)$ and all \mathcal{F} . In view of Proposition 3.5 we conclude that it suffices to check (3.2) for $\mathcal{G} = \mathbb{C}_S[\dim S]$, $S \in (\check{K} \backslash \check{\mathcal{B}})_{cl}$.

3.6. Pointwise purity and reduction to a property of the map on cohomology. Fix $\mathcal{F} \in \mathcal{L}$, $S \in (\check{K} \backslash \check{\mathcal{B}})_{cl}$ and let $i_S : S \hookrightarrow \check{\mathcal{B}}$ be the embedding. Then, by adjunction,

$$Hom_{\mathcal{L}}(\mathcal{F}, \mathbb{C}_S) = Ext_{D_{\check{K}}(S)}(i_S^*(\mathcal{F}), \mathbb{C}_S).$$

We have

$$(3.3) \quad \text{the complex } i_S^*(\mathcal{F}) \text{ is semi-simple.}$$

To see this, one has to work in the category of mixed sheaves. This can be done directly in this setting by utilizing Saito's mixed Hodge modules [S1, S2] or, by passing to the case of a finite base field and using ℓ -adic sheaves. In either case, the sheaf \mathcal{F} is pure. It follows from the arguments in [LV] that \mathcal{F} is actually *pointwise pure*, i.e., for all $x \in X$ the sheaves $i_x^* \mathcal{F}$ and $i_x^! \mathcal{F}$ are pure; here $i_x : \{x\} \rightarrow X$ denotes the inclusion. As S is an orbit of \check{K} , the complex $i_S^*(\mathcal{F})$ has constant cohomology sheaves, and its stalks are pure because of pointwise purity of \mathcal{F} . Thus $i_S^*(\mathcal{F})$ is a pure complex and hence it is semi-simple. As $i_S^*(\mathcal{F})$ is semi-simple, it is a direct sum of sheaves \mathbb{C}_S that have been shifted to various degrees. From this we conclude immediately that

$$Ext_{D_{\check{K}}(S)}(i_S^*(\mathcal{F}), \mathbb{C}_S) = Hom_{H_{\check{K}}^*(S)}(R\Gamma_{\check{K}}(i_S^*(\mathcal{F})), H_{\check{K}}^*(S)),$$

and hence that

$$(3.4) \quad \text{Hom}_{\mathcal{L}}(\mathcal{F}, \mathbb{C}_S) = \text{Hom}_{H_{\check{K}}^*(S)}(R\Gamma_{\check{K}}(i_S^*(\mathcal{F})), H_{\check{K}}^*(S)).$$

Recall now that we are attempting to show that the canonical map

$$\text{Hom}_{\mathcal{L}}(\mathcal{F}, \mathbb{C}_S) \rightarrow \text{Hom}_{\mathcal{C}_{oh}(\mathfrak{B})}(R\Gamma_{\check{K}}(\mathcal{F}), H_{\check{K}}^*(S))$$

is an isomorphism. Hence, by (3.4), it suffices to show that the canonical map

$$(3.5) \quad \text{Hom}_{H_{\check{K}}^*(S)}(R\Gamma_{\check{K}}(i_S^*(\mathcal{F})), H_{\check{K}}^*(S)) \rightarrow \text{Hom}_{\mathcal{C}_{oh}(\mathfrak{B})}(R\Gamma_{\check{K}}(\mathcal{F}), H_{\check{K}}^*(S)),$$

which is induced by the map $R\Gamma_{\check{K}}(\mathcal{F}) \rightarrow R\Gamma_{\check{K}}(i_S^*(\mathcal{F}))$ on global cohomology, is an isomorphism. Let us now write down this map in more concrete terms.

It is clear from Lemma 3.3 that the map in (3.5) can be written as

$$\text{Hom}_{\mathcal{C}_{oh}(\mathfrak{B}_w)}(R\Gamma_{\check{K}}(i_S^*(\mathcal{F})), \mathcal{O}_{\mathfrak{B}_w}) \rightarrow \text{Hom}_{\mathcal{C}_{oh}(\mathfrak{B})}(R\Gamma_{\check{K}}(\mathcal{F}), \mathcal{O}_{\mathfrak{B}_w}),$$

where $w = \phi(S)$ in the notations of Claim 2.1 and Lemma 3.3.

Using adjunction on the right hand side, we are reduced to showing that

$$(3.6) \quad \text{Hom}_{\mathcal{C}_{oh}(\check{\mathfrak{c}})}(R\Gamma_{\check{K}}(i_S^*(\mathcal{F})), \mathcal{O}_{\check{\mathfrak{c}}}) \rightarrow \text{Hom}_{\mathcal{C}_{oh}(\check{\mathfrak{c}})}(\mathcal{O}_{\check{\mathfrak{c}}} \otimes_{\mathcal{O}_{\mathfrak{B}}} R\Gamma_{\check{K}}(\mathcal{F}), \mathcal{O}_{\check{\mathfrak{c}}})$$

is an isomorphism.

Finally, let us recall that the map in the above formula is induced by the canonical map of coherent sheaves

$$(3.7) \quad \mathcal{O}_{\check{\mathfrak{c}}} \otimes_{\mathcal{O}_{\mathfrak{B}}} R\Gamma_{\check{K}}(\mathcal{F}) \rightarrow R\Gamma_{\check{K}}(i_S^*(\mathcal{F})).$$

3.7. Small codimension argument and localization to fixed points. Let us observe that the map (3.6) is obtained from (3.7) by duality, i.e., by homing into the structure sheaf. Thus, we conclude:

The map in (3.6) is an isomorphism if

- (3.8) a. the support of the kernel of (3.7) has positive codimension and
 b. the support of its cokernel has codimension two or higher.

We will now argue this. To do so we will make use of localization of equivariant cohomology for torus actions. Thus, we will shift to equivariant cohomology with respect to the torus \check{C} . We recall that, in general,

$$H_{\check{K}}^*(X, \mathcal{F}) = H_{\check{C}}^*(X, \mathcal{F})^{W(\check{K})};$$

here X is a variety and \mathcal{F} is a \check{K} -equivariant sheaf in $D_{\check{K}}(X)$.

Thus, in order to argue (3.8) for the map (3.7) it suffice to argue it for the map

$$(3.9) \quad H_{\check{C}}^*(S) \otimes_{H_{\check{C}}^*(\mathfrak{B})} H_{\check{C}}^*(\check{\mathfrak{B}}, \mathcal{F}) \rightarrow H_{\check{C}}^*(S, i_S^*(\mathcal{F})).$$

Let us recall the localization theorem for equivariant cohomology. We view $H_{\check{C}}^*(X, \mathcal{F})$ as an $H_{\check{C}}^*(\text{pt}, \mathbb{C}) = \mathbb{C}[\check{c}]$ module. Let $c \in \check{\mathfrak{c}}$. Then we have:

$$\mathcal{O}_{\check{\mathfrak{c}}, c} \otimes_{\mathcal{O}_{\check{\mathfrak{c}}}} H_{\check{C}}^*(X, \mathcal{F}) \cong \mathcal{O}_{\check{\mathfrak{c}}, c} \otimes_{\mathcal{O}_{\check{\mathfrak{c}}}} H_{\check{C}}^*(X^c, \mathcal{F}),$$

where $\mathcal{O}_{\check{\mathfrak{c}}, c}$ stands for the local ring at c .

Furthermore, taking the quotient by the maximal ideal we get an isomorphism on the fibers:

$$H_C^*(X, \mathcal{F})_c \cong H_C^*(X^c, \mathcal{F})_c,$$

We will apply the localization theorem in this form.

Let us recall that $Z_{\check{\mathfrak{g}}}(\check{c}) = \check{\mathfrak{t}}$ is a maximal torus in $\check{\mathfrak{g}}$. Recall also that the torus $\check{\mathfrak{t}}$ has elements which are regular in $\check{\mathfrak{g}}$, i.e., elements that do not lie in any root hyperplane for $\check{\mathfrak{g}}$. Let us first consider such a $c \in \check{\mathfrak{t}}$.

Then by the above localization theorem for equivariant cohomology we get isomorphisms of algebras and restrictions as follows:

$$(3.10) \quad \begin{array}{ccc} H_C^*(\check{\mathcal{B}})_c & \xrightarrow{\cong} & \bigoplus_{x \in \check{\mathcal{B}}^c} \mathbb{C}_x \\ \downarrow & & \downarrow \\ H_C^*(S)_c & \xrightarrow{\cong} & \bigoplus_{x \in S^c} \mathbb{C}_x. \end{array}$$

Furthermore, we get:

$$(3.11) \quad \begin{array}{ccc} R\Gamma_{\check{C}}(\mathcal{F})_c & \xrightarrow{\cong} & \bigoplus_{x \in \check{\mathcal{B}}^c} \mathcal{F}_x \\ \downarrow & & \downarrow \\ R\Gamma_{\check{C}}(i_S^* \mathcal{F})_c & \xrightarrow{\cong} & \bigoplus_{x \in S^c} \mathcal{F}_x. \end{array}$$

In the above formulas \mathcal{F}_x stands for the stalk of the complex \mathcal{F} at the point x . The isomorphisms and restrictions in (3.11) are compatible with the actions of the algebras in (3.10). Thus, we conclude that for c regular in $\check{\mathfrak{g}}$ we have

$$(H_C^*(S) \otimes_{H_C^*(\check{\mathcal{B}})} H_C^*(\check{\mathcal{B}}, \mathcal{F}))_c \cong H_C^*(S, i_S^*(\mathcal{F}))_c.$$

3.8. The small rank case. It remains to prove surjectivity in codimension one. Thus, we assume that $c \in \check{\mathfrak{t}} \cap H_\alpha$ is a general element; here $H_\alpha \subset \check{\mathfrak{h}}$ is the root hyperplane corresponding to the root α of $\check{\mathfrak{h}}$ in $\check{\mathfrak{g}}$. We write \check{C}_α for the codimension one sub torus of \check{C} corresponding to the root α . In other words, \check{C}_α is the torus generated by c .

Let us write Z'_c for the derived group of the centralizer of c in \check{G} . Then, by construction, the involution θ restricts to Z'_c and the group $(Z'_c)^\theta$ of fixed points of θ has rank 1. This imposes a very severe restriction on Z'_c . It can only happen¹ if Z'_c has rank one or if it has rank two and the Cartan involution is not inner. Thus Z'_c corresponds (up to isogeny) to one of the following real groups: $SL(2, \mathbb{R})$, $SU(2)$, $SL(2, \mathbb{C})$, $SL(3, \mathbb{R})$. It is also easy to see this from the geometric point of view. As $(Z'_c)^\theta$ has rank one and it has to have an open orbit on the flag manifold of Z'_c it follows that the flag manifold can have at most dimension three. Thus, we recover the list above. Furthermore, as we have assumed that \check{K} is connected, it follows that $(Z'_c)^\theta$ is connected and it is this form of the group we use. In the first two cases Z'_c is isomorphic to $SL(2)$ and in the remaining two to, respectively, $SL(2) \times SL(2)$ and $SL(3)$. The fixed point set $\check{\mathcal{B}}^c$

¹We thank David Vogan for pointing this out to us.

is the union of components each one of which is isomorphic to the flag variety of Z'_c . Therefore, the components of $\check{\mathcal{B}}^c$ are either \mathbb{P}^1 or $\mathbb{P}^1 \times \mathbb{P}^1$ or the quadric $Q \subset \mathbb{P}^2 \times \mathbb{P}^2$.

Recall that we have to show that for such a c we have surjection:

$$(H_C^*(S) \otimes_{H_C^*(\check{\mathcal{B}})} H_C^*(\check{\mathcal{B}}, \mathcal{F}))_c \twoheadrightarrow H_C^*(S, i_S^*(\mathcal{F}))_c.$$

We now proceed to reduce this statement to a statement about the group Z'_c . Let us write Z for a component of $\check{\mathcal{B}}^c$. Then, by localization theorem for equivariant cohomology, it suffices to show that

$$H_C^*(S \cap Z)_c \otimes_{H_C^*(Z)_c} H_C^*(Z, \mathcal{F})_c \twoheadrightarrow H_C^*(S \cap Z, i_S^*(\mathcal{F}))_c.$$

Furthermore, we can immediately reduce this to showing the surjectivity of

$$H_C^*(Z, \mathcal{F})_c \twoheadrightarrow H_C^*(S \cap Z, i_S^*(\mathcal{F}))_c,$$

and furthermore to the surjectivity of

$$(3.12) \quad H_{\check{C} \cap Z'_c}^*(Z, \mathcal{F}) \twoheadrightarrow H_{\check{C} \cap Z'_c}^*(S \cap Z, i_S^*(\mathcal{F})).$$

To this end we first claim that:

For $\mathcal{F} \in \mathcal{L}$ the complex $i_Z^*(\mathcal{F})$ is semi-simple.

As \mathcal{F} is pointwise pure, we conclude from lemma 3.6 below that $i_Z^*(\mathcal{F})$ is pure. Hence it is semi-simple.

We state the lemma using more general notation. Let X be the flag manifold of reductive algebraic group G and let C be a sub torus of a maximal torus T of G . Then:

Lemma 3.6. *Let $\mathcal{F} \in D_C(X)$ be pure. The its restriction $\mathcal{F}|_{X^C}$ to the fixed point set X^C of C is also pure.*

Proof. The centralizer $Z_G(C)$ is a connected reductive group which acts transitively on the components of X^C . Let us write Z for one of the components. As T acts on Z , let us consider a point $z \in Z$ which is fixed by T . Let us consider the root space decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(G, T)} \mathfrak{g}_\alpha.$$

the choice of the point z picks a particular choice of positive roots $\Phi(G, T)_z^+$. Let us now consider the tangent space $T_z X$. We can perform the following identifications:

$$T_z X = \bigoplus_{\alpha \in \Phi(G, T)_z^-} \mathfrak{g}_\alpha \quad \text{and} \quad T_z Z = \bigoplus_{\substack{\alpha \in \Phi(G, T)_z^- \\ \alpha|_C = 0}} \mathfrak{g}_\alpha.$$

Let us now choose $\mathbb{G}_m \subset G$ in such a way that $\alpha|_{\mathbb{G}_m}$ is positive for all $\alpha \in \Phi(G, T)_z^-$ for which $\alpha|_C \neq 0$. Then $X^{\mathbb{G}_m} = X^C$ and the \mathbb{G}_m action is attractive near Z . It is obviously attractive near z , but then it is attractive everywhere because $Z_G(C)$ acts transitively on Z and commutes with \mathbb{G}_m .

Of course \mathcal{F} is \mathbb{G}_m equivariant. By [Br], theorem 2, hyperbolic localization preserves purity. As our action is attracting, hyperbolic localization reduces to ordinary restriction.

□

We conclude that it suffices to prove (3.12) with \mathcal{F} replaced by an irreducible $\check{K} \cap Z'_c$ -equivariant sheaf \mathcal{G} on Z'_c , i.e., that

$$H_{\check{C} \cap Z'_c}^*(Z, \mathcal{G}) \twoheadrightarrow H_{\check{C} \cap Z'_c}^*(S \cap Z, i_S^*(\mathcal{G})).$$

We will prove the statement above by checking it case by case. Recall that Z'_c is one of the following groups: $SL(2, \mathbb{R})$, $SU(2)$, $SL(2, \mathbb{C})$, $SL(3, \mathbb{R})$. In each of these cases the closures Z'_c -orbits on Z are smooth and hence the sheaf \mathcal{G} is a constant sheaf. Thus, we are reduced to showing that in these cases

$$H_{\check{C} \cap Z'_c}^*(Z, \mathbb{C}) \twoheadrightarrow H_{\check{C} \cap Z'_c}^*(S \cap Z, \mathbb{C}).$$

Thus we are reduced to proving the following proposition which we state in a notation we only use here.

Proposition 3.7. *Let G be a semisimple algebraic group with an involution θ such that the fixed point set K of θ is connected and of real rank one. Let us write Z for the flag manifold of G , S for a closed K -orbit and C for a Cartan of K . Then we have a surjection*

$$H_C^*(Z, \mathbb{C}) \twoheadrightarrow H_C^*(S, \mathbb{C}).$$

Proof. In the case of $SU(2)$ we have $S \cap Z = Z$, so there is nothing to prove.

To deal with the other cases, we observe that proceeding as in the proof of 3.1 we see that

$$H_C^*(Z, \mathbb{C}) = \mathbb{C}[\mathfrak{t}] \otimes_{\mathbb{C}[\mathfrak{t}]^W} \mathbb{C}[\mathfrak{c}];$$

here \mathfrak{t} is the centralizer of \mathfrak{c} in \mathfrak{g} , a Cartan in \mathfrak{g} .

In all these cases the set $Z \cap S$ is a flag manifold of K and we can think of the map $S \cap Z \subset Z$ as an inclusion $K/K \cap B \subset G/B$ for a particular Borel B in G .

$$H_C^*(S, \mathbb{C}) = \mathbb{C}[\mathfrak{c}] \otimes_{\mathbb{C}[\mathfrak{c}]^{W(K)}} \mathbb{C}[\mathfrak{c}].$$

and thus the map is clearly a surjection. □

3.9. The case of disconnected \check{K} . Let us now consider the general case, i.e., the case when \check{K} is possibly not connected. We define the block variety \mathfrak{B} using the group \check{K}° , i.e.,

$$\mathfrak{B} = \check{\mathfrak{c}}/W(\check{K}^\circ, \check{\mathfrak{c}}) \times_{\check{\mathfrak{h}}/W} \check{\mathfrak{h}}$$

We note that we have the following exact sequence

$$1 \rightarrow W(\check{K}^\circ, \check{\mathfrak{c}}) \rightarrow W(\check{K}, \check{\mathfrak{c}}) \rightarrow \mathfrak{S} \rightarrow 1,$$

where $\mathfrak{S} = \check{K}/\check{K}^\circ$. Furthermore, recall again that $Z_{\check{\mathfrak{g}}}(\check{\mathfrak{c}}) = \check{\mathfrak{t}}$ is a maximal torus in $\check{\mathfrak{g}}$ and that the torus $\check{\mathfrak{c}}$ contains elements which are regular in $\check{\mathfrak{g}}$, i.e., elements that do not lie in any root hyperplane for \mathfrak{g} . Thus, $N_{\check{G}}(\check{\mathfrak{t}}) = N_{\check{G}}(\check{\mathfrak{c}})$. This implies that $W(\check{K}, \check{\mathfrak{c}}) \subset W(\check{G}, \check{\mathfrak{c}}) \subset W(\check{G}, \check{\mathfrak{t}}) = W$. This implies that

We have an action of \mathfrak{S} on $\check{\mathfrak{c}}/W(\check{K}^\circ, \check{\mathfrak{c}})$ such that

the induced action on the quotient $\check{\mathfrak{c}}/W(\check{G}, \check{\mathfrak{c}})$ is trivial.

Thus, the action of \mathfrak{S} on \mathfrak{B} takes place on the factor $\check{\mathfrak{c}}/W(\check{K}^\circ, \check{\mathfrak{c}})$.

We define a functor from \mathcal{D} to $\mathcal{C}oh_{\Gamma}(\mathfrak{B})$ by sending $\mathcal{F} \in \mathcal{D}$ to $R\Gamma_{\check{K}^{\circ}}(\mathcal{F})$. The case of connected \check{K} immediately implies the general case:

Theorem 3.8. *The functor $R\Gamma_{\check{K}^{\circ}} : \mathcal{L} \rightarrow \mathcal{C}oh^{\mathfrak{S}}(\mathfrak{B})$ is a full embedding.*

3.10. The description of the principal block. We summarize the results of this section in the following

Theorem 3.9. *Let $\mathcal{A} \subset \mathcal{C}oh(\mathfrak{B})$ be the full subcategory generated by the structure sheaves of components of \mathfrak{B} under the action of the functors $pr_{\alpha}^* pr_{\alpha*}$, direct sums and direct summands. Let $\mathcal{A}^{\mathfrak{S}}$ be the full subcategory in $\mathcal{C}oh^{\mathfrak{S}}(\mathfrak{B})$ defined as the preimage of \mathcal{A} under the forgetful functor $\mathcal{C}oh^{\mathfrak{S}}(\mathfrak{B}) \rightarrow \mathcal{C}oh(\mathfrak{B})$.*

Then the functor $R\Gamma_{\check{K}}$ induces an equivalence

$$\mathcal{L} \cong \mathcal{A}^{\mathfrak{S}},$$

where \mathcal{L} is defined in section 3.1.

Proof. Assume first that \check{K} is connected. Then the functor $R\Gamma_{\check{K}}|_{\mathcal{L}}$ is fully faithful by Theorem 3.2. Its image is generated under the functors $pr_{\alpha}^* pr_{\alpha*}$ by the images of constant sheaves on closed orbits in view of Lemma 3.4 and Proposition 3.5. These images are exactly the structure sheaves of components of \mathfrak{B} by Lemma 3.3. This proves the Theorem for connected \check{K} .

Turning to the general case, we see that the functor is fully faithful by Theorem 3.8. We claim that an object $\mathcal{F} \in \mathcal{C}oh^{\mathfrak{S}}(\mathfrak{B})$ lies in the image of $R\Gamma_{\check{K}}|_{\mathcal{L}}$ iff $\text{Forg}(\mathcal{F}) \in \mathcal{C}oh(\mathfrak{B})$ lies in the image of $R\Gamma_{\check{K}^{\circ}}|_{\mathcal{L}^{\circ}}$, where the notation is the self-explanatory. The "only if" part of the statement is clear from the commutative diagram

$$\begin{array}{ccc} \mathcal{L}_{\check{K}} & \xrightarrow{Res_{\check{K}^{\circ}}^{\check{K}}} & \mathcal{L}_{\check{K}}^{\circ} \\ R\Gamma_{\check{K}} \downarrow & & \downarrow R\Gamma_{\check{K}^{\circ}} \\ \mathcal{C}oh^{\mathfrak{S}}(\mathfrak{B}) & \xrightarrow{\text{Forg}} & \mathcal{C}oh(\mathfrak{B}) \end{array}$$

The "if" part follows by considering adjoint to functors in the last diagram and observing that $\mathcal{F} \in \mathcal{L}_{\check{K}}$ is a direct summand in $Ind_{\check{K}^{\circ}}^{\check{K}} \circ Res_{\check{K}^{\circ}}^{\check{K}}(\mathcal{F})$. \square

4. INTEGRAL REGULAR BLOCKS FOR A QUASI-SPLIT GROUP

4.1. Statement of results. In this section we provide a description of a block \mathcal{M} in the category of (\mathfrak{g}, K) modules with a regular integral generalized infinitesimal character for a pair (\mathfrak{g}, K) coming from a quasi-split real group $G_{\mathbb{R}}$.

It will be convenient to enlarge the category \mathcal{M} to the category $\widehat{\mathcal{M}}$ of pro-objects in \mathcal{M} . Every irreducible object in \mathcal{M} admits a projective cover in $\widehat{\mathcal{M}}$ which is unique up to an isomorphism.

Let $\mathcal{P} \subset \widehat{\mathcal{M}}$ be the full subcategory formed by finite sums of projective covers of irreducible (equivalently, any) objects in \mathcal{M} .

Let $\widehat{\mathfrak{B}}$ denote the completion of \mathfrak{B} at zero, by which we understand the spectrum of the completion of the coordinate ring $\mathcal{O}(\mathfrak{B})$ at the corresponding maximal ideal.

Define a full subcategory $\hat{\mathcal{A}} \subset \mathcal{Coh}(\hat{\mathfrak{B}})$ as the image of the category \mathcal{A} introduced in Theorem 3.9 under the completion functor $\mathcal{Coh}(\mathfrak{B}) \rightarrow \mathcal{Coh}(\hat{\mathfrak{B}})$. Likewise, $\hat{\mathcal{A}}^\mathfrak{S} \subset \mathcal{Coh}^\mathfrak{S}(\hat{\mathfrak{B}})$ is defined as the essential image of $\mathcal{A}^\mathfrak{S}$.

The goal of this section is the following

Theorem 4.1. *There exists a canonical equivalence $\mathcal{P} \cong \hat{\mathcal{A}}^\mathfrak{S}$.*

The equivalence arises from the following statements.

Along with \mathcal{M} we will need to consider the singular category of Harish-Chandra bimodules \mathcal{M}_{sing} . To define it assume first that G is simply connected, thus K is connected. Then we have an irreducible G -module with highest weight $\lambda + \rho$ which can be used to define translation functor $T_{\lambda \rightarrow -\rho}$ from the category of (\mathfrak{g}, K) -modules with generalized infinitesimal character λ to the category of (\mathfrak{g}, K) -modules with generalized infinitesimal character $-\rho$. We write \mathcal{M}_{sing} for the Serre subcategory generated by the image of \mathcal{M} under $T_{\lambda \rightarrow -\rho}$.

We now drop the assumption that G is simply connected. Let G_{sc} be the universal cover of G and let K_{sc} be the preimage of K under the universal covering map $G_{sc} \rightarrow G$.

Recall that the finite group $Z(G) \cap K$ acts on all modules in \mathcal{M} by a fixed character, which we denote by χ . Let $\tilde{\chi}$ be the pull back of χ to the group $Z(G_{sc}) \cap K_{sc}$. The pull back functor from (\mathfrak{g}, K) -modules to (\mathfrak{g}, K_{sc}) -modules is fully faithful and $Z(G_{sc}) \cap K_{sc}$ acts on modules in the image by $\tilde{\chi}$.

Set $\tilde{\chi}' = \tilde{\chi} \cdot \chi_\lambda^{-1}$ where χ_λ is the character by which $Z(G_{sc}) \cap K_{sc}$ acts on the irreducible G_{sc} module of highest weight $V_{\lambda+\rho}$.

Consider the category of (\mathfrak{g}, K_{sc}) modules with generalized infinitesimal character $-\rho$, where $Z(G_{sc}) \cap K_{sc}$ acts by the character $\tilde{\chi}'$. The translation functor $T_{\lambda \rightarrow -\rho}$ sends \mathcal{M} to that category; we let \mathcal{M}_{sing} denote the Serre subcategory generated by the image of \mathcal{M} under $T_{\lambda \rightarrow -\rho}$.

We let \mathcal{P}_{sing} be the category of projective pro-objects in \mathcal{M}_{sing} which are finite sums of projective covers of irreducible objects. Let us write $\mathcal{Coh}_0^\mathfrak{S}(\mathfrak{a}^*/W'_\mathcal{M})$ for the category of \mathfrak{S} -equivariant coherent sheaves on $\mathfrak{a}^*/W'_\mathcal{M}$ set theoretically supported at zero. We denote by $\widehat{\mathfrak{a}^*/W'_\mathcal{M}}$ the completion of $\mathfrak{a}^*/W'_\mathcal{M}$ at zero and then write $\mathcal{Coh}_{fr}^\mathfrak{S}(\widehat{\mathfrak{a}^*/W'_\mathcal{M}})$

Proposition 4.2. *We have canonical equivalences: $\mathcal{M}_{sing} \cong \mathcal{Coh}_0^\mathfrak{S}(\mathfrak{a}^*/W'_\mathcal{M})$ and*

$\mathcal{P}_{sing} \cong \mathcal{Coh}_{fr}^\mathfrak{S}(\widehat{\mathfrak{a}^/W'_\mathcal{M}})$. Here \mathcal{Coh}_0 denotes the category of coherent sheaves set theoretically supported at zero, $\widehat{\mathfrak{a}^*/W'_\mathcal{M}}$ stands for completion at zero, and \mathcal{Coh}_{fr} denotes the category of projective (equivalently, free) coherent sheaves.*

Recall the extended enveloping algebra $\tilde{U}(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \text{Sym}(\mathfrak{h})$, where the action of the center of the enveloping algebra $Z(\mathfrak{g})$ on $\text{Sym}(\mathfrak{h})$ comes from the Harish Chandra isomorphism, see, for example, [BeGi].

Let $\tilde{\mathcal{M}}_{sing}$ be the category of $(\tilde{U}(\mathfrak{g}), K)$ -modules, such that restricting the action of $\tilde{U}(\mathfrak{g})$ to $U(\mathfrak{g})$ one gets a module in \mathcal{M}_{sing} . We will call $\tilde{\mathcal{M}}_{sing}$ the extended singular block.

Corollary 4.3. *The extended singular block $\tilde{\mathcal{M}}_{sing}$ is naturally equivalent to $\mathcal{Coh}_0^\mathfrak{S}(\mathfrak{B})$.*

The arguments in section 1 of [BeGi] shows that the translation functor $T = T_{\lambda \rightarrow -\rho}$ admits a natural lifting to the “extended translation functor” $\tilde{T} : \mathcal{M} \rightarrow \tilde{\mathcal{M}}_{sing}$. If we write $res : \tilde{\mathcal{M}}_{sing} \rightarrow \mathcal{M}_{sing}$ for the restriction functor then

$$T = res \circ \tilde{T}.$$

In view of the Corollary it can be viewed as a functor $\tilde{T} : \mathcal{M} \rightarrow \mathcal{Coh}_0^{\mathfrak{S}}(\mathfrak{B})$. It clearly extends to the category of pro-objects and thus we obtain a functor $\tilde{T} : \mathcal{P} \rightarrow \mathcal{Coh}(\widehat{\mathfrak{B}})$.

Proposition 4.4. *The functor $\tilde{T}|_{\mathcal{P}}$ is fully faithful.*

Example 4.5. Before describing the last ingredient of the proof we give an example of an object in \mathcal{P} . The easiest one to describe is the projective cover of an irreducible $L \in \mathcal{M}$ supported on the whole of X . Thus $L = j_{!*}(\mathcal{L})$ where j is the embedding of the open $K \times H$ orbit X_0 into X and \mathcal{L} is an irreducible $K \times H$ equivariant local system on X_0 .

Let \mathcal{E} be the free pro-unipotent local system on the torus $A = X_0/K$ (cf. e.g. [?]). Then $\Pi_{\mathcal{L}} = j_!(\mathcal{L} \otimes \mathcal{E})$ will be called *the deformed principal series module* corresponding to \mathcal{L} . It is easy to see that $\Pi_{\mathcal{L}}$ is the projective cover of L .

Recall that we have fixed a bijection between $W^{\theta}/W_{\mathcal{M}}$ and the set of $K \times H$ -equivariant local system on X_0 such that $j_{!*}(\mathcal{L}) \in \mathcal{M}$. For $v \in W^{\theta}/W_{\mathcal{M}}$ let \mathcal{L}_v be the corresponding local system and L_v be the corresponding irreducible object.

Our next goal is to describe the image of the fully faithful functor \tilde{T} . To that and we recall the notion of wall-crossing functors using [BeGi] as a general reference. For a simple root α let $R_{\alpha} : \mathcal{M} \rightarrow \mathcal{M}$ be the corresponding *wall-crossing functor*. Recall that $R_{\alpha} = T_{\mu \rightarrow \lambda} \circ T_{\lambda \rightarrow \mu}$ where μ is a (generic) weight on the α -wall. The translation functors $T_{\lambda \rightarrow \mu}$ are defined in the same manner as the $T_{\lambda \rightarrow -\rho}$ introduced earlier.

Also, recall that components of \mathfrak{B} are in bijection with $W^{\theta}/W'_{\mathcal{M}}$, let \mathfrak{B}_v be the component corresponding to $v \in W^{\theta}/W'_{\mathcal{M}}$ and that in subsection 3.3 we introduced the notation $W_{\alpha} = \{1, s_{\alpha}\} \subset W$.

Proposition 4.6. *a) \mathcal{P} is generated under the action of the functors R_{α} and taking direct sums and summands by the deformed principal series modules $\Pi_{\mathcal{L}}$, where \mathcal{L} runs over the set of $K \times H$ equivariant local systems on the open orbit which belong to the block.*

b) We have a natural isomorphism

$$\tilde{T} \circ R_{\alpha} \cong pr_{\alpha}^* pr_{\alpha*} \circ \tilde{T}.$$

Here we identified the target category of \tilde{T} with $\mathcal{Coh}_0(\mathfrak{B})$ and use the notation

$$pr_{\alpha} : \mathfrak{B} = \mathfrak{a}^*/W'_{\mathcal{M}} \times_{\mathfrak{h}^*/W} \mathfrak{h}^* \rightarrow \mathfrak{a}^*/W'_{\mathcal{M}} \times_{\mathfrak{h}^*/W} \mathfrak{h}^*/W_{\alpha}.$$

c) If G is adjoint (so $W_{\mathcal{M}} = W'_{\mathcal{M}}$) then \tilde{T} sends $\Pi_{\mathcal{L}_v}$ to $\widehat{\mathcal{O}_{\mathfrak{B}_v}}$.

For a general G the functor \tilde{T} sends $\Pi_{\mathcal{L}_v}$ to $\bigoplus_{\tilde{v}} \widehat{\mathcal{O}_{\mathfrak{B}_{\tilde{v}}}}$, where \tilde{v} runs over the set of elements in $W^{\theta}/W'_{\mathcal{M}}$ projecting to $v \in W^{\theta}/W_{\mathcal{M}}$; the sheaf $\bigoplus_{\tilde{v}} \widehat{\mathcal{O}_{\mathfrak{B}_{\tilde{v}}}}$ is equipped with the obvious \mathfrak{S} -equivariant structure.

Proof. We check the geometric equivalent form of statement (a). Recall that, according to [BeGi], the wall-crossing functor R_α corresponds in the geometric language to the functor of convolving by the tilting module associated to the simple root α . Now proof proceeds in the same manner as the geometric proof of Casselman's submodule theorem in [BB]. Let us consider the projective cover P_L of the irreducible object L in \mathcal{M} . There exists a sequence of simple roots $\alpha_1, \dots, \alpha_m$, such that $R_{\alpha_1} \circ \dots \circ R_{\alpha_m}(L)$ has nonzero restriction to the open orbit in X . We see this as follows. Using induction on the dimension of support of L , we reduce to showing that for every irreducible $L \in \mathcal{M}$ with $\text{supp}(L) \subsetneq X$, there exists a simple root α , such that $\text{supp}(R_\alpha(L)) \supsetneq \text{supp}(L)$. It is easy to see that this is the case when $\dim(\text{supp}(L)) = \dim(\text{supp}(\pi_\alpha(L)))$, which obviously does happen for some simple root α .

It follows from the argument above that there is an \mathcal{L} such that $\text{Hom}(\Pi_{\mathcal{L}}, R_{\alpha_1} \circ \dots \circ R_{\alpha_m}(L)) \neq 0$. Thus, by adjunction, $\text{Hom}(R_{\alpha_m} \circ \dots \circ R_{\alpha_1}(\Pi_{\mathcal{L}}), L) \neq 0$. Therefore P_L embeds in the projective $R_{\alpha_m} \circ \dots \circ R_{\alpha_1}(\Pi_{\mathcal{L}})$ proving (a).

Part (b) follows from a general property of translation functors proved in [BeGi, Proposition 3.5]: the functor $T_{\lambda \rightarrow \mu} \circ T_{\mu \rightarrow \lambda}$ is isomorphic to the functor $N \mapsto \tilde{U}(\mathfrak{g}) \otimes_{\tilde{U}(\mathfrak{g})} W_\alpha N = \text{Sym}(\mathfrak{h}) \otimes_{\text{Sym}(\mathfrak{h})} W_\alpha N$ (In the notation of [BeGi] W_α is called W_μ). Now,

$$\begin{aligned} \tilde{T} \circ R_\alpha(M) &= \tilde{T}_{\lambda \rightarrow -\rho} \circ T_{\mu \rightarrow \lambda} \circ T_{\lambda \rightarrow \mu}(M) = \tilde{T}_{\mu \rightarrow -\rho} \circ T_{\lambda \rightarrow \mu} \circ T_{\mu \rightarrow \lambda} \circ T_{\lambda \rightarrow \mu}(M) \\ &= \tilde{T}_{\mu \rightarrow -\rho}(\text{Sym}(\mathfrak{h}) \otimes_{\text{Sym}(\mathfrak{h})} W_\alpha T_{\lambda \rightarrow \mu}(M)) = \text{Sym}(\mathfrak{h}) \otimes_{\text{Sym}(\mathfrak{h})} W_\alpha \tilde{T}_{\mu \rightarrow -\rho} T_{\lambda \rightarrow \mu}(M) \\ &= pr_\alpha^* pr_{\alpha*} \circ \tilde{T}(M) \end{aligned}$$

Part (c) will be checked in section 4.4 after the proof of Proposition 4.2.

4.2. Fully faithful translation to $\mathcal{M}_{\text{sing}}$: proof of Proposition 4.4. We have to check that

$$(4.1) \quad \text{Hom}(P, Q) \xrightarrow{\sim} \text{Hom}(\tilde{T}(P), \tilde{T}(Q))$$

for $P, Q \in \mathcal{P}$.

First we claim that (4.1) holds when either P or Q is of the form $T_{-\rho \rightarrow \lambda}(M)$ for some $M \in \mathcal{M}_{-\rho}$. The argument is very general in that we do not make use of the fact that P, Q or M are projective. Let us assume that $P = T_{-\rho \rightarrow \lambda}(M)$. By [BeGi, Proposition 3.5] we see that the functor $\tilde{T} \circ T_{-\rho \rightarrow \lambda} : \mathcal{M}_{\text{sing}} \rightarrow \tilde{\mathcal{M}}_{\text{sing}}$ coincides with the induction functor $\tilde{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})}$ and hence

$$\text{Hom}(\tilde{T}(T_{-\rho \rightarrow \lambda}(M)), \tilde{T}(Q)) = \text{Hom}(\tilde{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M, \tilde{T}(Q)).$$

Now, by adjunction properties we get

$$\begin{aligned} \text{Hom}(\tilde{U}(\mathfrak{g}) \otimes_{U(\mathfrak{g})} M, \tilde{T}(Q)) &= \text{Hom}(M, \text{res}(\tilde{T}(Q))) = \\ &= \text{Hom}(M, T(Q)) = \text{Hom}(T_{-\rho \rightarrow \lambda}(M), Q). \end{aligned}$$

The argument in the case when $Q = T_{-\rho \rightarrow \lambda}(M)$ is essentially the same.

We now turn to the general case. It suffices to prove the statement for a projective generator Q . Furthermore, it is straightforward to check that it suffices show that for the projective generator Q we can find an exact sequence of the form

$$0 \rightarrow Q \rightarrow T_{-\rho \rightarrow \lambda} M_1 \rightarrow T_{-\rho \rightarrow \lambda} M_2.$$

In fact we claim that we obtain such a sequence by setting $M_1 = T_{\lambda \rightarrow -\rho}(Q)$, $M_2 = T_{\lambda \rightarrow -\rho}(C)$, where $C = \text{coker}(Q \rightarrow T_{-\rho \rightarrow \lambda}(M_1))$. The maps are the canonical adjunction arrows. One sees easily that it suffices to show that the maps

$$(4.2a) \quad Q \rightarrow T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(Q)$$

and

$$(4.2b) \quad C \rightarrow T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(C)$$

are injective.

It suffices to prove these claims for a projective generator of \mathcal{M} . We can construct such a projective generator as follows. Let us write by $\mathcal{Q} = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V$, where V is finite dimensional K -representation, $U(\mathfrak{g})$ acts naturally from the left and K acts diagonally: $k(u \otimes v) = \text{Ad}(k)(u) \otimes kv$. The module \mathcal{Q} is a projective (\mathfrak{g}, K) -module (no conditions on the central character) as it represents the exact functor $\text{Hom}_K(V, \cdot)$ on K -types. We get a projective generator of \mathcal{M} by choosing V so that it includes at least one K -type from each irreducible module in \mathcal{M} and then setting $Q = \mathcal{Q}_{\hat{\lambda}}$ to be the formal completion of \mathcal{Q} at λ . More precisely we should take the direct summand of the completion which belongs to the block \mathcal{M} , but there is no harm arguing with the whole completion.

By a result of Kostant and Rallis, see [KR, Theorem 15] and also [BBG, Theorem 1.6], \mathcal{Q} is a free module over the algebra $\text{Sym}(\mathfrak{p})^{\mathfrak{k}} = \text{Sym}(\mathfrak{a})^{W_{\mathbb{R}}}$. Furthermore, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on \mathcal{Q} via the projection $\text{Sym}(\mathfrak{h})^W \rightarrow \text{Sym}(\mathfrak{a})^{W_{\mathbb{R}}}$, i.e., via the closed subscheme $\mathfrak{a}^*/W_{\mathbb{R}} \subset \mathfrak{h}^*/W$. In particular, $Q = \mathcal{Q}_{\hat{\lambda}}$ is torsion free over $\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\hat{\lambda}}$. Note also that the considerations above show that the center $Z(\mathfrak{g})$ acts on any finitely generated (\mathfrak{g}, K) -module via the quotient $\text{Sym}(\mathfrak{a})^{W_{\mathbb{R}}}$, i.e., the category of finitely generated (\mathfrak{g}, K) -modules is supported on the closed subscheme $\mathfrak{a}^*/W_{\mathbb{R}} \subset \mathfrak{h}^*/W$.

We have the following:

Lemma 4.7. *If $M \in \mathcal{M}$ is torsion free over $\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\hat{\lambda}}$ then $M \rightarrow T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(M)$ is injective.*

Proof. Let us write $\mathcal{K}_{\hat{\lambda}}$ for the fraction field of $\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\hat{\lambda}}$. Then because M is torsion free we have $M \subset M_{\mathcal{K}_{\hat{\lambda}}} = M \otimes_{\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\hat{\lambda}}} \mathcal{K}_{\hat{\lambda}}$. But $M_{\mathcal{K}_{\hat{\lambda}}}$ is semisimple and

$$M_{\mathcal{K}_{\hat{\lambda}}} \text{ is a direct summand of } T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(M_{\mathcal{K}_{\hat{\lambda}}})$$

and, of course,

$$T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(M_{\mathcal{K}_{\hat{\lambda}}}) = T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(M) \otimes_{\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\hat{\lambda}}} \mathcal{K}_{\hat{\lambda}}$$

Thus, $M \rightarrow T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(M)$ is injective. \square

Thus, as Q is torsion free over $\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\hat{\lambda}}$ we conclude by the lemma that (4.2a) is injective. Thus, to show the injectivity of (4.2b) it suffices to show that C is torsion free over $\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\hat{\lambda}}$. Concretely, we have to show that for $f \in \mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\hat{\lambda}}$ (which we of course assume not to be a unit) the map $C/fC \rightarrow T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(C/fC)$ is injective. By the 5-lemma this reduces to showing that the map $Q/fQ \rightarrow T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(Q/fQ)$ is injective. Furthermore, by Nakayama's lemma, it suffices to check the injectivity on

the level of the fiber $\bar{Q} = Q \otimes_{\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_{\bar{\lambda}}} \mathbb{C}$ of Q/fQ at λ , i.e., that $\bar{Q} \rightarrow T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(\bar{Q})$ is injective.

The module \bar{Q} has finite length. If $\bar{Q} \rightarrow T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}(\bar{Q})$ is not injective then its kernel is a nonzero submodule $M \subset \bar{Q}$ such that $T_{\lambda \rightarrow -\rho}(M) = 0$. It is a general fact that if $T_{\lambda \rightarrow -\rho}(M) = 0$ then the dimension of the support variety of M , the Gelfand–Kirillov dimension of M , is less than dimension of the K -nilpotent cone $\mathcal{N} \cap \mathfrak{p}$. One can see this easily, for example from the geometric description of the that category of $U(\mathfrak{g})$ -modules at $-\rho$. However, using the canonical filtration of \mathcal{Q} generated by $1 \otimes V$ over $U(\mathfrak{g})$, we see that $gr(\bar{Q}) = V \otimes (Sym(\mathfrak{p}) \otimes_{\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_0} \mathbb{C})$ as a $Sym(\mathfrak{g})$ -module. Therefore $gr(\bar{Q})$ is a Cohen-Macaulay module with maximum dimensional support as $Sym(\mathfrak{p}) \otimes_{\mathcal{O}(\mathfrak{a}^*/W_{\mathbb{R}})_0} \mathbb{C}$ is a complete intersection. Thus $gr(\bar{Q})$ has no submodules with a smaller dimension of support, which shows that $M = 0$.

4.3. Cross action and intertwining functors. We will use the *cross action* introduced in [Vo2, Definition 8.3.1] on the set of irreducible objects in \mathcal{M} , denoted by $w : L \mapsto w \times L$ and its relation to *intertwining functors* $I_w : D^b(\mathcal{M}) \rightarrow D^b(\mathcal{M})$ of [BB]. Recall that, as λ is dominant, the latter can be characterized by:

$$R\Gamma_{w \cdot \lambda}(I_w(M)) \cong R\Gamma_{\lambda}(M),$$

where $\Gamma_{\mu}(M)$ denotes the direct summand in $\Gamma(X, M)$ on which the abstract Cartan \mathfrak{t} acts through the generalized character μ .

For an irreducible object $L \in \mathcal{M}$ let us write Δ_L for the standard cover, i.e., the ! extensions of the corresponding local system on a K orbit in X . Let ∇_L denote the costandard hull of L , i.e. the $*$ extensions of the corresponding local system on a K orbit in X .

We will also need some properties of the intertwining functors I_w .

Proposition 4.8. *a) Let μ be such that $(\mu + \rho, \alpha) = 0$ for a simple root α . Then we have a canonical isomorphism*

$$T_{\lambda \rightarrow \mu}(M) \cong T_{\lambda \rightarrow \mu}(I_{s_{\alpha}}(M)).$$

b) We have an action of W on $K^0(\mathcal{M})$ given by $w : [M] \mapsto [I_w(M)]$. We have

$$K^0(T_{-\rho \rightarrow \lambda} T_{\lambda \rightarrow -\rho}) = \sum (-1)^{\ell(w)} w \in \mathbb{Z}[W].$$

Proof. These statements are well-known properties of intertwining functors which hold for all D -modules (not just for K -equivariant ones). \square

Cross action is closely related to intertwining functors. We are primarily interested in local systems on the orbits attached to maximally split Cartan when these relation is easier to state.

For an irreducible object $L \in \mathcal{M}$ let O_L denote the open orbit in the support of L .

Proposition 4.9. *Assume that O_L is attached to the maximally split Cartan.*

a) We have an equality in the Grothendieck group $K^0(\mathcal{M})$: $[I_w(\Delta_L)] = [\Delta_{w \times L}]$.

b) Assume that L is supported on the open orbit. Then the object $I_w(\Delta_L)$ is a perverse sheaf whose support coincides with that of $w \times L$, and we have:

$$I_w(\Delta_L)|_{O_{w \times L}} \cong (w \times L)|_{O_{w \times L}}.$$

c) Let μ be such that $(\mu + \rho, \alpha) = 0$ for a simple root α . Then we have canonical isomorphisms:

$$T_{\lambda \rightarrow \mu}(\Delta_{s_\alpha \times L}) \cong T_{\lambda \rightarrow \mu}(\Delta_L).$$

Proof. a) For the maximally split torus in a quasi-split group every simple root is either complex or real of type II. Then one of the formulas (b1), (b2), (c2) or (e) in [Vo3, Definition 6.4] applies. These formulas relate the cross action of a simple reflection to the operator T_s ; also, according to [LV] the operator $-T_s$ coincides with the effect of I_s on the K -group. In fact, [Vo3] is concerned with the action of these operators on q -deformed version of the K -group acted upon by the Hecke algebra, to pass to our present setting one needs to specialize the variable u appearing in [Vo3, §6.4] to one; notice also that the basis elements γ considered in *loc. cit.* correspond to classes of the form $(-1)^d[\Delta_L]$, where d is the dimension of support of L , also the automorphism of the K -group induced by the functor I_s . Taking into account that for a real type II root and $s \notin \tau$ (where we follow notations of *loc. cit.*) one has $s \times \gamma = \gamma$, see [Vo2, Proposition, 8.3.18], we get the statement.

b) In view of (a) it suffices to check that $I_w(\Delta_L)$ is a perverse sheaf. The open K -orbit in \mathcal{B}_0 is the quotient of K by $K \cap T$ for a maximal torus $T \subset G$, thus it is affine. This shows that $I_w(\Delta_L)$ is perverse, since it can be presented both as a direct image and a direct image with compact support of a perverse sheaf under an affine morphism, cf. e.g. [BM, §5.1].

To check (c) consider the projection $\pi_\alpha : G/B \rightarrow G/P_\alpha$. The following cases are possible:

- i) The map $\pi_\alpha|_O$ is an \mathbb{A}^1 fibration (complex root). Then it is easy to see that $I_{s_\alpha}(\Delta_L) \cong \Delta_{s_\alpha \times L}$, so the needed isomorphism follows from Proposition 4.8(a).
- ii) The map $\pi_\alpha|_O$ is an \mathbb{C}^* fibration (real root) and the restriction of L to such a fiber is nontrivial. Then again $I_{s_\alpha}(\Delta_L) \cong \Delta_{s_\alpha \times L}$, so there is nothing to prove.
- iii) The map $\pi_\alpha|_O$ is an \mathbb{C}^* fibration and the restriction of L to such a fiber is nontrivial. Then the object $I_{s_\alpha}(\Delta_L)$ can be described as follows: $I_{s_\alpha}(\Delta_L) = j'_! j''_*(L)$, where $j'' : O \rightarrow \pi_\alpha^{-1}\pi_\alpha(O)$ and $j' : \pi_\alpha^{-1}\pi_\alpha(O) \rightarrow G/B$ are the embeddings. Also $s_\alpha \times (L) = L$ in this case. Thus we have a canonical map $I_{s_\alpha}(\Delta_L) \rightarrow \Delta_L$ whose kernel and cokernel are isomorphic to $j'_!(L')$ where L' is the extension of L to $\pi_\alpha^{-1}\pi_\alpha(O)$. Since $T_{\lambda \rightarrow \mu}$ kills that latter object, we get the desired isomorphism.

Notice that the possibility of $\pi_\alpha|_O$ being either a two-to-one map (type I real root situation) or a \mathbb{P}^1 -fibration (compact root situation) is excluded by the assumption that O is attached to the maximally split Cartan while G is quasisplit. \square

Recall the deformed principal series modules Π_L introduced in Example 4.5. The same definition applies for any irreducible L .

Corollary 4.10. *Let L be as in Proposition 4.9. Let μ be such that $(\mu + \rho, \alpha) = 0$ for a simple root α . Then we have canonical isomorphisms:*

$$T_{\lambda \rightarrow \mu}(\Pi_{s_\alpha \times L}) \cong T_{\lambda \rightarrow \mu}(\Pi_L).$$

This isomorphism commutes with the action of $\widehat{\mathcal{O}(\mathfrak{a}^)}$ via the natural action of W^θ on \mathfrak{a} .*

4.4. The singular block: proof of Proposition 4.2. We start by recalling some standard facts.

Lemma 4.11. *a) The extended translation functor is a Serre factorization.*

b) $T_{\lambda \rightarrow -\rho}$ sends irreducible to irreducible or zero, every irreducible at $-\rho$ comes from a unique irreducible at λ .

Proof. (b) follows from (a) and (a) follows directly from [BeGi]. \square

Lemma 4.12. *Assume that G is adjoint. There exists exactly one irreducible object L_0 in \mathcal{M} which does not go to zero under $T_{\lambda \rightarrow -\rho}$.*

Proof. This follows from the main result of [Vo4, Theorem 1.15]. It is easy to see that the bijection between irreducible objects in \mathcal{M} and $\check{\mathcal{M}}$ sends a module satisfying the property in the statement of the Lemma into a finite dimensional module. It is clear that a block in the category of $(\check{\mathfrak{g}}, \check{K})$ -modules can not contain more than one finite dimensional module as \check{G} is simply connected and therefore \check{K} is connected. Thus the Lemma follows. \square

Remark 4.13. Another, more elementary result of Vogan [Vo1, Theorem 6.2] provides a direct classification for irreducible modules of maximal Gelfand-Kirillov dimension; it is easy to see that this condition is equivalent to nonvanishing of the image of the module under $T_{\lambda \rightarrow -\rho}$. It would be interesting to deduce the Lemma directly from that classification.

Until section 4.5 we assume that G is adjoint.

Lemmas 4.11, 4.12 show that there exists exactly one irreducible object in $\mathcal{M}_{\text{sing}}$. Let $P \in \widehat{\mathcal{M}}_{\text{sing}}$ be its projective cover. To prove Proposition 4.2 in the present case it suffices to construct an isomorphism $\text{End}(P) \cong \widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}}$.

Consider the deformed principal series module $\Pi_{L, \hat{\rho}} \in \mathcal{M}_{\text{sing}}$ in the singular block defined by: $\Pi_{L, \hat{\rho}} = \Gamma(X_0^{sc}, L \otimes \mathcal{E})_{\hat{\rho}}$. Here X_0^{sc} is the open $K^{sc} \times T^{sc}$ orbit in $X^{sc} := G^{sc}/U$. The local system L is the pull-back of a K^{sc} -equivariant local system on G/B belonging to the block. The local system \mathcal{E} is the free pro-unipotent K^{sc} -equivariant local system on G/N_0 . Finally, the subscript denotes the maximal direct summand where the commutative subalgebra $\text{Sym}(\mathfrak{h}) \subset D(G/U)$ acts through the completion at the singular central character $-\rho$.

Observe that the commutative ring $\widehat{\mathcal{O}(\mathfrak{a}^*)}$ acts naturally on \mathcal{E} , hence it acts on $\Pi_{L, \hat{\rho}}$.

The following key statement implies Proposition 4.2.

Lemma 4.14. *a) The action of $\widehat{\mathcal{O}(\mathfrak{a}^*)}$ on $\text{Hom}_{\mathcal{M}_{\text{sing}}}(\Pi_{L, \hat{\rho}}, P)$ makes it a free rank one module. Here $\widehat{\mathcal{O}(\mathfrak{a}^*)}$ acts on $\text{Hom}_{\mathcal{M}_{\text{sing}}}(\Pi_{L, \hat{\rho}}, P)$ via its action on $\Pi_{L, \hat{\rho}}$.*

b) We have $\Pi_{L, \hat{\rho}} \cong P^{|W_{\mathcal{M}}|}$.

c) The group $W_{\mathcal{M}}$ acts on $\Pi_{L, \hat{\rho}}$, this action commutes with the natural action of $\widehat{\mathcal{O}(\mathfrak{a}^)}$ via the natural action of $W_{\mathcal{M}}$ on $\widehat{\mathcal{O}(\mathfrak{a}^*)}$.*

Proof. Let Q denote the projective cover of the irreducible $L_0 \in \mathcal{M}$ (introduced in Lemma 4.12).

Using that $\Pi_{L, \hat{\rho}} = T_{\lambda \rightarrow -\rho}(\Pi_{L, \hat{\lambda}})$, $T_{-\rho \rightarrow \lambda}(P) = Q$ and adjointness between $T_{\lambda \rightarrow -\rho}$, $T_{-\rho \rightarrow \lambda}(P)$, we see that to check (a) it suffices to see that $j^*(Q)$ is the direct sum of irreducible local systems on the open orbit belonging to the block, tensored with the free prounipotent local system (here j stands for the embedding of the preimage of the open K orbit into X).

This follows from the result of [Ch], [Vo1] by a particular case of BGG reciprocity. Namely, it follows from *loc. cit.* that the dual principal series object $j_*(\mathcal{L})$ contains L_0 in its Jordan-Hoelder series with multiplicity one; here \mathcal{L} is an irreducible local system on X_0 which belongs to the block. On the other hand, for such \mathcal{L} we have:

$$\text{Ext}^{>0}(j^*(Q), \mathcal{L}) = \text{Ext}^{>0}(Q, j_*(\mathcal{L})) = 0,$$

which implies that $j^*(Q)$ is projective. Thus $j^*(Q) \cong \bigoplus \mathcal{L}_i \otimes \mathcal{E}^{d_i}$, for some $d_i \in \mathbb{Z}_{\geq 0}$, where \mathcal{L}_i runs over the set of local systems on X_0 belonging to the block. Now we have

$$d_i = \dim \text{Hom}(j^*Q, \mathcal{L}_i) = \dim \text{Hom}(Q, j_*(\mathcal{L}_i)) = [L_0 : j_*(\mathcal{L}_i)] = 1.$$

This proves (a).

Because of exactness and adjointness of translation functors they send projective (pro)objects to projective ones. Since $\Pi_{L, \hat{\rho}} = T_{\lambda \rightarrow -\rho}(\Pi_{L, \hat{\lambda}})$ and $\Pi_{L, \hat{\lambda}}$ is projective, it follows that $\Pi_{L, \hat{\rho}}$ is projective. Since $\mathcal{M}_{\text{sing}}$ has a unique irreducible object with projective cover P , we see that $\Pi_{L, \hat{\rho}} \cong P^d$ for some d .

We have

$$\begin{aligned} d &= \dim \text{Hom}(T_{\lambda \rightarrow -\rho}(\Pi_{L, \hat{\lambda}}), T_{\lambda \rightarrow -\rho}(L_0)) = \\ (4.3) \quad &\dim \text{Hom}(\Pi_{L, \hat{\lambda}}, T_{-\rho \rightarrow \lambda}T_{\lambda \rightarrow -\rho}(L_0)) = [j!_*(\mathcal{L}[\dim X]) : T_{-\rho \rightarrow \lambda}T_{\lambda \rightarrow -\rho}(L_0)] = \\ &[j!_*(\mathcal{L}[\dim X]) : T_{-\rho \rightarrow \lambda}T_{\lambda \rightarrow -\rho}(j!(\mathcal{L}[\dim X]))], \end{aligned}$$

where the last equality follows from the isomorphism $T_{\lambda \rightarrow -\rho}(L_0) \cong T_{\lambda \rightarrow -\rho}(j!(\mathcal{L}[\dim X]))$, which is a consequence of the result of [Ch], [Vo1] mentioned above. (Here we used notation $[L : M]$ for the multiplicity of an irreducible L in the Jordan-Hoelder series of an object M).

By Proposition 4.8(a) we have an equality in the Grothendieck group $K^0(\mathcal{M})$:

$$[T_{-\rho \rightarrow \lambda}T_{\lambda \rightarrow -\rho}(j!(\mathcal{L}[\dim X]))] = \sum M_i,$$

where M_i runs over the set of principal series modules coming from orbits attached to the maximally split Cartan, each one appearing with multiplicity $|W_{\mathcal{M}}|$. (Recall the running assumption that G is adjoint, so $W_{\mathcal{M}} = W'_{\mathcal{M}}$). Since the irreducible object $j!_*(\mathcal{L}[\dim X])$ appears once in the Jordan-Hoelder series of a principal series module coming from the open orbit and does not appear in the other principal series modules, formula (4.3) shows that $d = |W_{\mathcal{M}}|$, which yields statement (b).

Finally, to show (c) we need to produce for every $w \in W_{\mathcal{M}}$ an automorphism of $\Pi_{L, \hat{\rho}} = T_{\lambda \rightarrow -\rho}(\Pi_{L, \hat{\lambda}})$. It is obtained as a composition

$$T_{\lambda \rightarrow -\rho}(\Pi_{L, \hat{\lambda}}) \cong T_{\lambda \rightarrow -\rho}(\Pi_{w \times L, \hat{\lambda}}) \cong T_{\lambda \rightarrow -\rho}(\Pi_{L, \hat{\lambda}}),$$

where the first isomorphism comes from Corollary 4.10, while the last one amounts to the condition $w \in W_{\mathcal{M}}$. \square

Lemma 4.14 implies Proposition 4.2. It is clear from Lemma 4.14(b) that $\widehat{\mathcal{O}(\mathfrak{a}^*)} \cong \text{Hom}_{\mathcal{M}_{\text{sing}}}(\Pi_{L, \hat{\rho}}, P)$ is a $\text{End}(P) - \text{End}(\Pi_{L, \hat{\rho}})$ bimodule satisfying the second commutator property; by this we mean that each of the two rings acts faithfully and coincides with the commutant of the other ring. Thus Lemma 4.14(c) shows that $\text{End}(P) \hookrightarrow \widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}}$. It remains to check that this embedding is actually an isomorphism. In view of the second commutator property, it suffices to check that it becomes an isomorphism after base change to the field of fractions of $\widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}}$ (the latter is a commutative ring which lies in the center of $\text{End}(P)$, being a quotient of the center of the enveloping algebra). The localization $\text{End}(P)_{\text{loc}}$ is a subextension of the finite field extension $\text{Frac}(\widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}})/\text{Frac}(\widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}})$ (where Frac stands for the field of fraction). If it was a proper subextension we would have

$$\dim_{\text{End}(P)_{\text{loc}}}(\text{Hom}(\Pi_{L, \hat{\rho}}, P)_{\text{loc}}) < \dim_{\text{Frac}(\widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}})}(\text{Frac}(\widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}})) = |W_{\mathcal{M}}|,$$

while Lemma 4.14(b) shows that $\text{rank}_{\text{End}(P)_{\text{loc}}}(\text{Hom}(\Pi_{L, \hat{\rho}}, P)_{\text{loc}}) = |W_{\mathcal{M}}|$. \square

4.4.1. Proof of Proposition 4.6(c). The statement amounts to the following. For $v \in W^{\theta}/W_{\mathcal{M}}$ consider the space $V = \text{Hom}_{\mathcal{P}_{\text{sing}}}(P, \tilde{T}(\mathcal{P}_{\mathcal{L}_{v, \lambda}}))$. It carries an action of $\text{End}(\mathcal{P}_{\mathcal{L}_{v, \lambda}}) \cong \widehat{\mathcal{O}(\mathfrak{a}^*)}$; Lemma 4.14 implies that this action makes V a free rank 1 module over $\widehat{\mathcal{O}(\mathfrak{a}^*)}$. For $w \in W^{\theta}$ we can define an action of $\widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}}$ on V by twisting the above action of $\widehat{\mathcal{O}(\mathfrak{a}^*)}$ by w and restricting the above action from $\widehat{\mathcal{O}(\mathfrak{a}^*)}$ to $\widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}}$. On the other hand, V carries a natural action of $\text{End}(\mathcal{P}_{\text{sing}}) \cong \widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}}$. We need to check that the two actions of $\widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}}$ coincide provided that w maps to v under the projection $W^{\theta} \rightarrow W^{\theta}/W_{\mathcal{M}}$.

For $w = 1$ this follows from the construction of the isomorphism $\text{End}(\mathcal{P}_{\text{sing}}) \cong \widehat{\mathcal{O}(\mathfrak{a}^*)}^{W_{\mathcal{M}}}$. Now the case of an arbitrary w follows from Corollary 4.10. \square

4.5. Non-adjoint groups. We now complete the proof of Proposition 4.2 and Proposition 4.6(c) by reducing the general case to the case when the group G is adjoint. Recall that the category $\mathcal{M}_{\text{sing}}$ consists of (\mathfrak{g}, K_{sc}) -modules that we obtained by first viewing \mathcal{M} as (\mathfrak{g}, K_{sc}) -modules and then translating them to infinitesimal character $-\rho$. We can proceed in the same for the category \mathcal{M}^{ad} . Recall, however, as explained in subsection 2.3, that the category \mathcal{M}^{ad} has infinitesimal character $\lambda + \mu$. We can compose the translation of \mathcal{M} to $-\rho$ as $T_{\lambda \rightarrow -\rho} = T_{\lambda + \mu \rightarrow -\rho} \circ T_{\lambda \rightarrow \lambda + \mu}$. Therefore the categories $\mathcal{M}_{\text{sing}}$ and $\mathcal{M}_{\text{sing}}^{ad}$ are related in the following manner. We have a restriction functor

$$\text{Res}_{K_{sc}}^{(K_{ad})_{sc}} : \mathcal{M}_{\text{sing}}^{ad} \rightarrow \mathcal{M}_{\text{sing}} \text{ having an exact bi-adjoint } \text{Ind}_{K_{sc}}^{(K_{ad})_{sc}}$$

Our goal is to construct a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{sing}^{ad} & \xrightarrow{\cong} & Coh(\mathfrak{B}_0) \\ \text{Res}_{K_{sc}}^{(K_{ad})_{sc}} \downarrow & & \downarrow Av_{\mathfrak{S}} \\ \mathcal{M}_{sing} & \xrightarrow{\cong} & Coh^{\mathfrak{S}}(\mathfrak{B}_0) \end{array}$$

where $Av_{\mathfrak{S}}$ is the induction functor (adjoint to the forgetful functor $Coh^{\mathfrak{S}}(\mathfrak{B}_0) \rightarrow Coh(\mathfrak{B}_0)$). We have already constructed the equivalence in the top row. Each of the vertical arrows has an exact bi-adjoint and both categories on the bottom row can be described as the category of modules over the corresponding monad acting on the corresponding category in the top row. Thus we need only to check that the equivalence in the top row is compatible with the above monads. This reduces to the following.

Lemma 4.15. *The equivalence $\mathcal{M}_{sing}^{ad} \cong Coh(\mathfrak{B}_0)$ is naturally compatible with the action of the group \mathfrak{S} on the two categories. Here \mathfrak{S} acts on the left hand side by twisting a module with a character $\chi \in \mathfrak{S} \cong \tilde{K}_{ad}/K$, while the action on the right hand side comes from the action of $\mathfrak{S} \cong W_{\mathcal{M}}/W'_{\mathcal{M}}$ on \mathfrak{B} .*

Proof. We return to the situation of §2.2. We let $\widehat{\mathcal{L}_{\mathcal{M}}(X_0)}$, $\widehat{\mathcal{L}_{\mathcal{M}}^{ad}(X_0)}$ denote the formal neighborhood of $\mathcal{L}_{\mathcal{M}}(X_0)$ (respectively, $\mathcal{L}_{\mathcal{M}}^{ad}(X_0)$) in the space of K (resp. \tilde{K}_{ad}) equivariant T -monodromic local systems² on X_0 .

We have:

$$\widehat{\mathfrak{a}^*}/W'_{\mathcal{M}} = \widehat{\mathcal{L}_{\mathcal{M}}^{ad}(X_0)}/W^{\theta}.$$

The twisting action of abelian group \mathfrak{S} on the space of characters induces an action on the right hand side. To prove the Lemma we need to see that the latter action is compatible with the action of $\mathfrak{S} \cong W_{\mathcal{M}}/W'_{\mathcal{M}}$ on the left hand side.

We will use the general observation that two commuting actions of a finite abelian group on an algebraic variety (or on a formal completion of such) with the same quotient coincide. We transport the action on the left hand side to the right hand side and check that the two actions of \mathfrak{S} on $\widehat{\mathcal{L}_{\mathcal{M}}^{ad}(X_0)}/W^{\theta}$ commute and have the same quotient.

The space $\widehat{\mathcal{L}_{\mathcal{M}}^{ad}(X_0)}$ carries commuting actions of W^{θ} and \mathfrak{S} which induce the above two actions on $\widehat{\mathcal{L}_{\mathcal{M}}^{ad}(X_0)}/W^{\theta}$, thus we see that the two actions of \mathfrak{S} commute. Coincidence of the quotients by the two actions is clear from the equality

$$\widehat{\mathcal{L}_{\mathcal{M}}(X_0)} = \widehat{\mathcal{L}_{\mathcal{M}}^{ad}(X_0)}/\mathfrak{S}$$

which shows that

$$\widehat{\mathfrak{a}^*}/W_{\mathcal{M}} = (\widehat{\mathcal{L}_{\mathcal{M}}^{ad}(X_0)}/W^{\theta})/\mathfrak{S}.$$

□

²In more classical terms this can be described as the set of characters of the real torus $A_{\mathbb{R}}$.

4.6. The principal block for split groups. In this subsection we sketch an alternative approach to the proof of Proposition 4.4 in the special case when the group $G_{\mathbb{R}}$ is split and the block under consideration is the principal block. This material is not used in the rest of the paper, so the details are omitted.

In the case when the group is split and we consider the principal block, there is a concrete description of indecomposable projective modules P at the singular infinitesimal central character; this also yields a description of the projective cover of a *special* irreducible module at regular infinitesimal central character, namely it is isomorphic to $T_{-\rho \rightarrow \lambda}(P)$. Here by a special irreducible module we mean an irreducible module which survives translation to $-\rho$. Such irreducible modules in the principal block correspond to constant sheaves on special closed orbits; we call a closed K -orbit O on X *special* if for any simple root α the dimension of the image of O in the partial flag variety G/P_{α} equals $\dim O$. Another characterization of special orbits is as follows. Let us write $\pi : T^*X \rightarrow X$ for the projection and $\mu : T^*X \rightarrow \mathcal{N}$ for the moment map. The special orbits are precisely the closed K -orbits such that the image $\mu(T_O^*X)$ of the conormal bundle T_O^*X meets the regular locus \mathcal{N}_{reg} of \mathcal{N} . Conversely, if we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition then the union of special closed orbits is the closure of $\pi(\mu^{-1}(\mathfrak{p} \cap \mathcal{N}_{reg}))$.

To simplify the discussion, let us now also assume that K is connected. Given a special orbit O_s we write $i_s : O_s = K/B_K \hookrightarrow G/B$, we let L_s denote the irreducible module associated to the trivial local system on O_s . To this situation we can associate a weight $\rho_s = i_s^*(\rho_G) - 2\rho_K$ of K and consequently an irreducible K -representation V_s . We have

a) ρ_s is a dominant weight of K .

and

b) If $s' \neq s$ then $\rho_{s'}$ is not of the form $\rho_s + i_s^*(\lambda^+)$

where λ^+ is a sum of positive roots for G .

The representation V_s has the following properties:

- (1) The multiplicity of V_s in the irreducible module $T_{\lambda \rightarrow -\rho}(L_s)$ equals one, its multiplicity in any other irreducible (\mathfrak{g}, K) -module with infinitesimal singular central character $-\rho$ equals zero.
- (2) The representation V_s is *fine* in the sense of [BGG], i.e., its restriction to the group \mathfrak{S} of order two elements in the maximal torus $C \subset K$ is a direct sum of distinct characters and the group W_K permutes these characters transitively.

It follows that the projective cover of the irreducible module $T_{\lambda \rightarrow -\rho}(L_s)$ is given by $P_s = (U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V_s)_{-\rho}$, where the subscript denotes completion at the infinitesimal central character $(-\rho)$. This provides an alternative way to carry out one of the steps of the proof of Proposition 4.4. More precisely, we can deduce from the above description of P_s that $\text{End}(P_s)$ is a free module over the completion of $\mathcal{O}(\mathfrak{a}/W_{\mathbb{R}} = \mathfrak{t}/W)$ (recall that G is assumed to be split) of rank $|W/W_b|$. To this end observe that $\text{End}(U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V_s) = \text{Hom}_K(V_s, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} V_s)$. The right hand side admits a filtration whose associated graded is $\text{Hom}_K(V_s, \mathcal{O}(\mathfrak{p}) \otimes V_s)$, thus it suffices to see that the latter space is a free module over $\mathcal{O}(\mathfrak{t}/W)$ of rank $|W/W_b| = \dim(V_s)$.

The space $\text{Hom}_K(V_s, \mathcal{O}(\mathfrak{p}) \otimes V_s)$ can also be thought of as $\text{End}_{\mathcal{C}_{oh}^\kappa(\mathfrak{p})}(\mathcal{O}(\mathfrak{p}) \otimes V_s)$. Recall that $\mathfrak{p} = K(\mathfrak{a})$, which shows that $\text{End}_{\mathcal{C}_{oh}^\kappa(\mathfrak{p})}(\mathcal{O}(\mathfrak{p}) \otimes V_s)$ embeds into $\text{End}_{\mathcal{C}_{oh}(\mathfrak{a})}(\mathcal{O}(\mathfrak{a}) \otimes V_s)$; furthermore, the image is contained in the space of endomorphisms E whose action on the fiber at a point $x \in \mathfrak{a}$ commutes with the action of the centralizer $Z_K(x)$. It is not hard to check that the image is in fact equal to that space; moreover, the commutation with centralizer can be checked only for regular elements $x \in \mathfrak{a}$. For such an element the stabilizer is identified with \mathfrak{S} , and V_s splits as a sum of distinct characters of \mathfrak{S} due to V_s being fine. This shows that the generic rank of $\text{End}_{\mathcal{C}_{oh}^\kappa(\mathfrak{p})}(\mathcal{O}(\mathfrak{p}) \otimes V_s)$ as an $\mathcal{O}(\mathfrak{t}/W)$ -module equals $\dim(V_s) = |W/W_b|$, which implies the desired property of $\text{End}(P_s)$.

5. THE MAIN RESULT

We now put the descriptions in sections 3, 4 together to get a comparison between the two categories.

For a graded algebra A let A_{dg}^\bullet denote the corresponding differential graded algebra with zero differential; for a DG-algebra A^\bullet let $DG - \text{mod}(A^\bullet)$ denote the subcategory of perfect complexes in the derived category of differential graded modules over A^\bullet .

Theorem 5.1. *There exists a graded algebra A together with*

- i) an equivalence $\mathcal{M} \cong A - \text{mod}^{fl}$, where $A - \text{mod}^{fl}$ is category of A -modules of finite length.*
- ii) an equivalence $\mathcal{D} \cong DG - \text{mod}(A_{dg}^\bullet)$ sending irreducible perverse sheaves to direct summands of the free module.*

A more concrete reformulation of Theorem 5.1 is as follows.

Set $L = \oplus L_i$, where L_i runs over a set of representatives for isomorphism classes of irreducible objects in $\text{Perv}_K^0(\tilde{\mathcal{B}})$.

To deal with the quasisplit side it will be convenient to enlarge the category \mathcal{M} to the category $\widehat{\mathcal{M}}$ of pro-objects in \mathcal{M} . Every irreducible object in \mathcal{M} admits a projective cover in $\widehat{\mathcal{M}}$ which is unique up to an isomorphism. Let $P = \sum P_i$ where P_i runs over a set of representatives for isomorphism classes of indecomposable projectives in $\widehat{\mathcal{M}}$.

Theorem 5.2. *a) The differential graded algebra $R\text{Hom}_{\mathcal{D}}(L, L)$ is formal.*

b) The algebra $\text{End}(P)$ is isomorphic to the completion of the graded algebra $\text{Ext}^\bullet(L, L)$ with respect to the augmentation ideal topology.

Here part (a) is proven by a standard weight argument. Deduction of Theorem 5.1 from Theorem 5.2 is also standard. In the following sections we concentrate on the proof of (b).

We refer to e.g. [BGS] for a definition of a graded version of an abelian category. By a graded version of a triangulated category \mathcal{C} we understand another triangulated category $\tilde{\mathcal{C}}$ together with an autoequivalence S of $\tilde{\mathcal{C}}$ (the grading shift functor) denoted by $S : M \mapsto M(1)$ the "degrading" functor $d : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ and an isomorphism, $d \circ S \cong d$, such that for $M, N \in \tilde{\mathcal{C}}$ we have $\text{Hom}(d(M), d(N)) \cong \bigoplus_{\mathbb{Z}} \text{Hom}(\tilde{M}, \tilde{N}(n))$ and \mathcal{C} is generated as a triangulated category by the image of d .

Corollary 5.3. *There exist graded versions of categories \mathcal{M} , $\widehat{\mathcal{M}}$, \mathcal{D} and an equivalence $\kappa : D^b(\widehat{\mathcal{M}}^{gr}) \cong \mathcal{D}^{gr}$, such that*

- i) $\kappa(M(1)) = \kappa(M)(1)[1]$, where $M \mapsto M(1)$ denotes the shift of grading.*
- ii) κ sends indecomposable projective objects in $\widehat{\mathcal{M}}^{gr}$ to shifts of irreducible objects in $Perv_K^0(\check{\mathcal{B}})$.*

Proof. Define \mathcal{M}^{gr} to be the category of finite length (equivalently, finite dimensional) graded modules over the graded algebra A from Theorem 5.1. Define $\widehat{\mathcal{M}}^{gr}$ to be the category of finitely generated graded A modules. Then the equivalences of the Corollary follow from Theorem 5.1.

One can summarize the statement of the Corollary by saying that \mathcal{M} and \mathcal{D} are *Koszul dual* one to the other.

5.1. The representation theoretic description of the equivariant category.

The category \mathcal{D} can be described in module theoretic terms as follows. Consider the abelian category $\check{\mathcal{M}}$, the principal block of $(\check{\mathfrak{g}}, \check{K})$ modules with a fixed regular integral infinitesimal central character. Let $Q = \sum Q_i$ be a minimal projective generator in the category of pro-objects in that category, here Q_i are pairwise non-isomorphic indecomposable projectives. Denoting $\check{A} = End(Q)$ we get an equivalence $\check{\mathcal{M}} \cong \check{A} - mod^{fl}$. The action of the center of the enveloping algebra makes \check{A} a $Sym(\check{\mathfrak{h}})$ -algebra, where $\check{\mathfrak{h}}$ is the abstract Cartan of $\check{\mathfrak{g}}$. Consider the DG-algebra $\overline{\check{A}}^\bullet = \check{A} \otimes_{Sym(\check{\mathfrak{h}})}^L \mathbb{C}$. Using e.g. [B, §9.3] it is easy to check the following:

Proposition 5.4. *We have $\mathcal{D} \cong DG - mod^{fl}(\overline{\check{A}}^\bullet)$.*

To clarify the relation with the result of [So2] we need to present a slight modification of the above constructions.

Let $S \subset \mathfrak{h}^*$ be a vector subspace, we then get $S^\perp \subset \check{H}^* = \mathfrak{h}$.

One can deduce from Theorem 5.1 reinterpreted by means of Proposition 5.4 that the triangulated categories of DG-modules over the DG algebras $A \otimes_{Sym(\mathfrak{h})}^L \mathcal{O}(S)$ and $\check{A} \otimes_{Sym(\check{\mathfrak{h}})}^L \mathcal{O}(S^\perp)$ are in a Koszul duality relation described in Corollary 5.3.

Furthermore, assume that S is transversal to $\mathfrak{a}^* \subset \mathfrak{h}^*$, i.e. $Tor_{>0}^{\mathcal{O}(\mathfrak{h}^*)}(\mathcal{O}(S), \mathcal{O}(\mathfrak{a}^*)) = 0$. Then using the main result of [BBG] one can show that $Tor_{>0}^{Sym(\mathfrak{h})}(A, \mathcal{O}(S)) = 0$, thus the first of the two dual triangulated categories discussed in the previous paragraph is identified with $D^b(A_S - mod)$, where $A_S = A \otimes_{\mathcal{O}(\mathfrak{h}^*)} \mathcal{O}(S)$. If a similar Tor vanishing condition holds on the dual side, with A, S replaced by \check{A}, S^\perp , then the second category is $D^b(\check{A}_{S^\perp} - mod)$, $\check{A}_{S^\perp} = \check{A} \otimes_{\mathcal{O}(\mathfrak{h}^*)} \mathcal{O}(S^\perp)$. Notice that $A_S - mod^{fl}$, $\check{A}_{S^\perp} - mod^{fl}$ are the categories of Harish-Chandra modules for (\mathfrak{g}, K) , respectively $(\check{\mathfrak{g}}, \check{K})$, subject to a certain condition on the action of the center of $U\mathfrak{g}$.

A particularly nice situation occurs when both Tor vanishing conditions hold simultaneously. In this case Theorem 5.1 implies

The algebras A_S, \check{A}_{S^\perp} controlling the categories of Harish-Chandra modules for (\mathfrak{g}, K) , respectively $(\check{\mathfrak{g}}, \check{K})$, are dual Koszul quadratic algebras.

We mention two examples of that situation.

Koszul duality for category O . Let $G_{\mathbb{R}}$ be the complex group viewed as a real group, thus $G = H \times H$ for some reductive group H , $\check{G} = \check{H} \times \check{H}$ and $K = H \subset H \times H$, $\check{K} = \check{H} \subset \check{H} \times \check{H}$ are the diagonal subgroups. Let \mathfrak{t} denote the maximal torus of H . Then $\mathfrak{a}^* \subset \mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{t}^*$ is the anti-diagonal and similarly for ${}^L\mathfrak{a}^*$. Let $S = \mathfrak{t}^* \times \{0\} \subset \mathfrak{t}^* \oplus \mathfrak{t}^* = \mathfrak{h}^*$. Then both transversality conditions hold. The categories $A_S - \text{mod}$, $\check{A}_{S^\perp} - \text{mod}$ in this case are identified with the usual category O (with, say, a fixed integral regular central character and no Cartan diagonalizability condition). The equivalence from the previous paragraph reduces in this case to the one constructed in [So2].

Koszul duality for the principal block in a split group. Assume that \check{G} is split group, thus ${}^L\mathfrak{a}^* = \check{\mathfrak{h}}^*$. Then the second condition holds for any S . Consider $S = \mathfrak{a}^*$, thus $S^\perp = 0$, so we are back at a special case of the situation considered in the main body of the paper. We obtain the following

Corollary 5.5. *In the situation of Theorem 5.1, assume that $\check{G}_{\mathbb{R}}$ is split. Then there exist two Koszul dual graded rings $A, A^!$ with $A^!$ finite dimensional such that:*

$$\check{\mathcal{M}} \cong A^! - \text{mod}_{fg}, \check{\mathcal{D}} \cong D^b(A^! - \text{mod}_{fd}), \mathcal{M} \cong A - \text{mod}_{nilp}.$$

Notice that the above shows that $D_K(G/B) \cong D^b(\text{Perv}_K(G/B))$ when G is a split semi-simple group. It would be interesting to obtain a topological proof of this fact.

6. FURTHER DIRECTIONS: HODGE D -MODULES

We were lead to the idea of the block variety and the interpretation of the functor $\tilde{T} : \mathcal{M} \rightarrow \mathcal{Coh}_0^{\mathfrak{S}}(\mathfrak{B})$ from Hodge theoretic considerations. We briefly explain the idea.

The category \mathcal{M} has a mixed Hodge module version \mathcal{M}^{mix} . Let us write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the Cartan decomposition and let us recall that $T^*X/H = \tilde{\mathfrak{g}}$.

We write $\tilde{\mathfrak{p}}$ for the inverse image of \mathfrak{p} under the moment map $T^*X/H = \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$. We write $\mathcal{Coh}^{\mathbb{C}^* \times K}(\tilde{\mathfrak{p}})_0$ for the category of $\mathbb{C}^* \times K$ -equivariant coherent sheaves on $\tilde{\mathfrak{p}}$ set theoretically supported on $\tilde{\mathfrak{p}} \cap \tilde{\mathcal{N}}$, the union of conormal bundles of K -orbits on \mathcal{B} .

Let \mathcal{M}_{Ho} denote the full subcategory of the category of K -equivariant Hodge D -modules M on X , such that $\text{Forg}(M) \in \mathcal{M}$; here Forg denotes the functor of forgetting the Hodge structure.

We have a functor

$$\tilde{\text{gr}} : \mathcal{M}_{\text{Ho}} \longrightarrow \mathcal{Coh}^{\mathbb{C}^* \times K}(\text{Spec}(\mathcal{O}_{\tilde{\mathfrak{p}}}))_0$$

taking a Hodge D -module to its associated graded with respect to the Hodge filtration.

Conjecture 6.1. *a) There exists a full subcategory $\mathcal{M}^{\text{mix}} \subset \mathcal{M}_{\text{Ho}}$ which is a graded version of \mathcal{M} .*

b) For $M, N \in \mathcal{M}^{\text{mix}}$ we have

$$\text{Ext}^i(\text{Forg}(M), \text{Forg}(N)) \xrightarrow{\sim} \text{Ext}_{\mathcal{Coh}^K(\tilde{\mathfrak{p}})}^i(\text{gr}(M), \text{gr}(N))$$

for all i .

In the special case when $G_{\mathbb{R}}$ is a complex group the Conjecture will be established in [BR].

In order to relate this Conjecture to our present methods we need the following observation. Let $\Sigma \subset \mathfrak{p}$ be a transversal slice to a regular nilpotent orbit, thus Σ is contained in the set of regular elements \mathfrak{g}^{reg} . Let $\tilde{\Sigma}$ denote the set of pairs (x, χ) where $x \in \Sigma$ and χ is a character of the abelian algebraic group $Z_K(x)$, the centralizer of x in K .

It is not hard to see that $\tilde{\Sigma}$ is naturally equipped with the structure of an ind-algebraic variety, an infinite union of components, each one of which is a ramified covering of Σ . For $\mathcal{F} \in \mathcal{Coh}^K(\mathfrak{p})$ each fiber of the sheaf $\mathcal{F}|_{\Sigma}$ is graded by the set of characters of $Z_K(x)$; this observation can be upgraded to a definition of a coherent sheaf $\mathcal{F}_{\tilde{\Sigma}}$ whose direct image to Σ is identified with $\mathcal{F}|_{\Sigma}$. This clearly extends to a functor $\kappa : \mathcal{Coh}^K(\tilde{\mathfrak{p}}) \rightarrow \mathcal{Coh}(\tilde{\Sigma} \times_{\mathfrak{t}^*/W} \mathfrak{t}^*)$.

Note that for a regular semisimple element $x \in \Sigma$ the abelian group $Z_K(x)$ can be identified with the stabilizer of a point in the open K -orbit \mathcal{B}_0 .

Lemma 6.2. *a) Consider the set of pairs $(x, \chi) \in \tilde{\Sigma}$, where x is regular semisimple and χ is such that the corresponding K -equivariant, T -monodromic local system on X_0 belongs to \mathcal{M} . This is an open subset in a component $\tilde{\Sigma}_{\mathcal{M}} \subset \tilde{\Sigma}$.*

b) We have $\tilde{\Sigma} \cong \mathfrak{a}^/W_{\mathcal{M}}$.*

c) For $M \in \mathcal{M}_{\mathcal{H}_0}$ the coherent sheaf $\mathrm{gr}(M)_{\Sigma}$ is supported on $\tilde{\Sigma} \times_{\mathfrak{t}^/W} \mathfrak{t}^*$.*

Conjecture 6.3. *Let $\mathcal{M}^{\mathrm{mix}}$ be as in Conjecture 6.1. For $M, N \in \mathcal{M}^{\mathrm{mix}}$ we have a canonical isomorphism:*

$$\kappa \circ \mathrm{gr}(M) \cong \tilde{T}(\mathrm{Forg}(M)).$$

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